A characterisation of R_1-spaces via a Mal'tsev condition

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Introduction

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Categorical algebra studies algebras by working in a category satisfying certain axioms.

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- Mal'tsev and protomodular categories;
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One may ask: of which of these types of categories is a given category of spaces an example?

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For example, consider **Top**, and the notion of regular category.

Definition (Barr, Grillet and van Osdol, 1971)

A category $\mathbb{C}$ is called *regular* if it has finite limits and admits a pullback-stable (regular epi, mono)-factorisation system.

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Definition (Barr, Grillet and van Osdol, 1971)

A category $\mathbb{C}$ is called *regular* if it has finite limits and admits a pullback-stable (regular epi, mono)-factorisation system.

Examples of regular categories include **Set** and moreover, any variety of universal algebras. However, **Top** is *not* regular because quotient maps are not pullback-stable.

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- $\mathbf{Top}^{\mathrm{op}}$ is not a Mal'tsev category;
- however, it (and moreover, any regular category) has a largest full subcategory which is (in a certain sense);
- moreover, we show that the objects of this subcategory are characterised by a known separation axiom – they are the so-called R_1-spaces.

Mal'tsev varieties

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Theorem (Mal'tsev, 1954)

For a variety $\mathbb{X}$ of universal algebras, the following are equivalent:

- *the composition of congruences on any object in $\mathbb{X}$ is commutative;*
- *the algebraic theory of $\mathbb{X}$ contains a ternary term μ satisfying*

$$\mu(x, y, y) = x = \mu(y, y, x).$$

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For example, the variety of all groups satisfies this condition: take $\mu(x, y, z) = x - y + z$.

Naturally Mal'tsev categories

One way to generalize Mal'tsev varieties to categories is as follows:

Definition (Johnstone, 1977)

A category $\mathbb{C}$ is called a *naturally Mal'tsev category* if the identity functor $1_{\mathbb{C}}$ admits an internal Mal'tsev operation μ in the functor category $\mathrm{Fun}(\mathbb{C}, \mathbb{C})$.

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This is a very strong condition: for example, a pointed variety of universal algebras is a naturally Mal'tsev category if and only if it is abelian (i.e. it is the variety of R-modules for some ring R).

Other equivalent conditions

The conditions in the theorem are also equivalent to each of the following:

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- every reflexive internal relation is an equivalence relation (where by an *internal relation* we mean a relation from A to B which is a subalgebra of $A \times B$);
- every internal relation R from A to B is *difunctional*, i.e. it satisfies

$$(x_1 R y_2 \wedge x_2 R y_2 \wedge x_2 R y_1) \Rightarrow x_1 R y_1.$$

This second condition is due to Lambek (1957).

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Given any internal relation $r : R \to A \times B$ in a category $\mathbb{C}$, and an object S in $\mathbb{C}$, there is a corresponding relation on hom-sets:

$$\mathrm{Hom}(S, r) : \mathrm{Hom}(S, R) \to \mathrm{Hom}(S, A) \times \mathrm{Hom}(S, B)$$

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We call an internal relation *reflexive, transitive, symmetric, difunctional* ... when for every S, the relation above is *reflexive, transitive, symmetric, difunctional* ... in the usual set-theoretic sense.

Relations in categories

Notation: If we have an internal relation $R \to X \times Y$ and two morphism $x : S \to X$ and $y : S \to Y$, we will write xRy to mean that x and y are related by the relation

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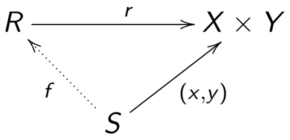
This is equivalent to the existence of a morphism f such that the diagram commutes:



Mal'tsev categories

Definition (Carboni, Lambek and Pedicchio, 1990)

A *Mal'tsev category* is a category in which every internal relation is difunctional.

That is, a Mal'tsev category is a category in which every object S in $\mathbb{C}$ satisfies:

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Question: Is $\mathbf{Top}^{\mathrm{op}}$ a Mal'tsev category?
Answer: No. One can construct a non-difunctional co-relation using the Sierpiński space.

Condition (D)

Recall the condition on the previous slide:

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Theorem

Let $\mathbb{C}$ be a regular category admitting binary coproducts. Then $D(\mathbb{C})$, the full subcategory of all objects satisfying condition (D), is the largest full subcategory of $\mathbb{C}$ which is Mal'tsev and which is closed under regular quotients and binary coproducts in $\mathbb{C}$.

Proof

Note that $D(\mathbb{C})$ is closed under regular quotients and binary coproducts because the Yoneda embedding $Y : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}^{\mathbb{C}}$ takes colimits to limits. Thus a colimit/regular quotient of objects in $D(\mathbb{C})$ takes each internal relation to a limit/regular subobject of difunctional relations, which will itself be difunctional.

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To show that $D(\mathbb{C})$ is the largest such category, we need the following lemma.

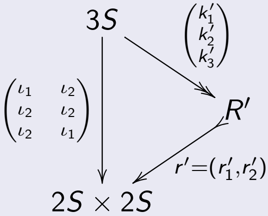
Proof

Lemma

For a regular category $\mathbb{C}$ admitting binary coproducts and an object S in $\mathbb{C}$, the following are equivalent:

(a) *S satisfies* (D)*;*

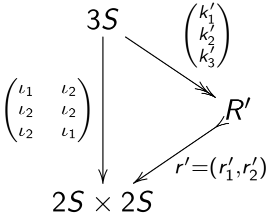
(b) *$\iota_1 R' \iota_1$, where $\iota_1 : S \to 2S$ is the first coproduct injection and (R', r_1', r_2') is the relation from $2S$ to $2S$ appearing in the (regular epi, mono)-factorisation of the vertical morphism in the following diagram:*



(1)

Proof

This lemma is adapted from a theorem due to Bourn and Z. Janelidze. One direction is easy to show: if we consider the relation R' below:



(2)

then we have $\iota_1 R' \iota_2$, $\iota_2 R' \iota_2$ and $\iota_2 R' \iota_1$, so $\iota_1 R' \iota_1$ if $\mathrm{Hom}(S, R')$ is difunctional.

Proof

Conversely, given a relation $R \to X \times Y$ for which $x_1 R y_2$, $x_2 R y_2$ and $x_2 R y_1$, consider the pullback.

$$\begin{array}{ccc} R'' & \xrightarrow{\quad p \quad} & R \\ {\scriptstyle r''=(r_1'',r_2'')}\Big\downarrow & & \Big\downarrow{\scriptstyle r=(r_1,r_2)} \\ 2S \times 2S & \xrightarrow[\binom{x_1}{x_2} \times \binom{y_1}{y_2}]{} & X \times Y \end{array} \tag{3}$$

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By construction of R'' we have that $\iota_1 R'' \iota_2$, $\iota_2 R'' \iota_2$ and $\iota_2 R'' \iota_1$, so that R'' contains the relation R' from the theorem.

Proof

Conversely, given a relation $R \to X \times Y$ for which $x_1 R y_2$, $x_2 R y_2$ and $x_2 R y_1$, consider the pullback.

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By construction of R'' we have that $\iota_1 R'' \iota_2$, $\iota_2 R'' \iota_2$ and $\iota_2 R'' \iota_1$, so that R'' contains the relation R' from the theorem. But then $\iota_1 R'' \iota_1$, and we obtain $x_1 R y_1$ as required.

Topological spaces

Let us return to **Top**^{op}. What we have shown is that:

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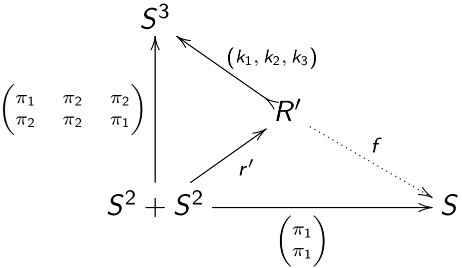
Topological spaces

Let us return to $\mathbf{Top}^{\mathrm{op}}$. What we have shown is that:

- $\mathbf{Top}^{\mathrm{op}}$ is not a Mal'tsev category,
- but it admits a largest full subcategory $D(\mathbf{Top}^{\mathrm{op}})$ which is;
- moreover, we can test to see if an object is in this subcategory, by checking for a relation between two particular morphisms – i.e. the existence of a certain morphism.

Test for the dual of (D)

Following the theorem, a space S satisfies the dual of (D) if and only if there is a continuous map f which makes the diagram commute:



where R' is the subspace of S^3 which is the image of the vertical map, i.e. the subspace

$$\{(x, y, z) \mid x = y \vee y = z\}$$

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The map f is uniquely defined if it exists, so it is enough to check continuity.

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Lemma

The map f is continuous if and only if S is an R_1-space in the sense of Davis (1961), i.e. satisfies the following separation axiom:

(R_1) for all $x, y \in X$, if there exists an open set A such that $x \in A$ and $y \notin A$, then there exists disjoint open sets B and C such that $x \in B$ and $y \in C$.

Proof

As a set map, $f : R' \to S$ takes a triple (x, y, z) to x if $y = z$ and to z if $x = y$.

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As a set map, $f : R' \to S$ takes a triple (x, y, z) to x if $y = z$ and to z if $x = y$.
($\Rightarrow$): Suppose f is continuous. Let A be an open set with $x \in A$ and $y \notin A$. Then

$$f^{-1}(A) = \{(x.y, z) \mid (x = y \wedge z \in A) \vee (y = z \wedge x \in A)$$

is open in R'. Thus there exist open sets U, W and V in S such that $(x, y, y) \in U \times W \times V \cap R' \subseteq f^{-1}(A)$. But (z, z, y) can never be in $f^{-1}(A)$ for any z – otherwise $y \in A$! Thus U and W do not intersect, and x and y are separated by neighbourhoods as required.

Proof

($\Leftarrow$): Suppose S is an R_1-space and let A be open in S with $(x, y, y) \in f^{-1}(A)$. If $y \in A$, then

$$(x, y, y) \in A \times A \times A \cap R' \subseteq f^{-1}(A).$$

Otherwise, pick open sets $x \in U$ and $y \in V$ such that $U \cap V = \varnothing$ and note that

$$(x, y, y) \in U \times V \times V \cap R' \subseteq f^{-1}(A)$$

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R_1-spaces

Where do R_1-spaces fit in?

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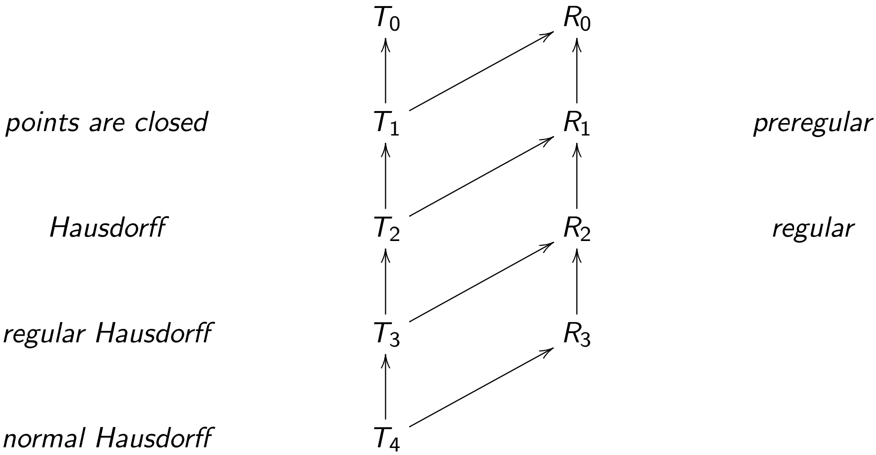

Concluding remarks

We have presented a connection between two notions:

- the notion of a Mal'tsev variety, or term, and
- the notion of an R_1-space.

via the language of categories.

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A natural question is: do other such connections exist?

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- Other term conditions from algebra can be translated this way: for example, the case for a binary subtraction term has been considered by Z. Janelidze (in fact that result predates this one).
- Work on finding relational conditions for other separation axioms is still in progress (e.g. R_0 and regular spaces).

Thank you.