Syllabus: European Talbot workshop on structured ring spectra

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Background reading for spectra [EKMM97, HSS00, Schw, Le91] and for K-theory [DGM13, We13].

1. **Mentor's talk** Overview talk: What are structured ring spectra? What are they good for? In which respects do they behave differently from ordinary rings?

2. **Thom spectra and Bousfield localization** What is Bousfield localization and what effect does it have [Ra84]? Describe some local phenomena and examples (Lubin-Tate, Morava K-theory as quotient, $K(n)$-local sphere spectrum), Morava stabilizer group ([Ra86]). Thom spectra as an important family of examples of ring spectra ('classical' construction [Mi, R98, Ko96, Ra84, M77, M09b], other constructions [Schl09, ABGHR]).

3. **Infinite loop space machines and K-theory** Every permutative category gives rise to a connective spectrum and conversely every connective spectrum can be modelled by a permutative category. One can jazz up the construction to allow bimonoidal categories as input and produce ring spectra; bipermutative categories giving rise to commutative ring spectra. Explain the results and give an overview which structured ring spectra arise in this way [ShSh79, S74, M10, EM09, M78, M09a, T95].

4. **(Dg and simplicial) rings and structured ring spectra** How do (simplicial, dg) rings embed into structured ring spectra? Schwede [Schw99] proved that connective $H\mathbb{Z}$-algebras correspond to simplicial rings, Shipley showed that differential graded rings are Quillen equivalent to $H\mathbb{Z}$-algebras. Earlier Dundas showed that every functor with smash product can be approximated by simplicial rings [D97]. There are pitfalls, however: Dugger-Shipley found $H\mathbb{Z}$-algebras that are equivalent as ring spectra but whose corresponding dg-rings are not quasiisomorphic and Lawson showed that commutative Γ-rings are not what you might think they are [DS07, L09].

5. **Picard and Brauer groups** Given a commutative ring spectrum R, we would like to understand its modules and algebras, in particular its invertible module spectra and possible Azumaya algebra spectra over it. Explain some results about Picard groups, define the Brauer group of a commutative ring spectrum and present some examples [MaSt, St92, HMS94, HS99, Ka10, AG14, BR05, BRS12].

6. **Galois extensions of structured ring spectra** Discuss the case of ordinary commutative rings, define Galois extension of commutative ring spectra and explain some of the properties. Examples: $KO \to KU$, $K(n)$-local examples between the $K(n)$-local sphere and Lubin-Tate, cochain Galois extensions between S and $H\mathbb{F}_p$ [R08]. How does one determine brave new Galois groups [Ma]?

7. **Hochschild and topological Hochschild homology** Briefly recall Hochschild homology for algebras [Lo98] and then explain the various definitions of THH, the Bökstedt spectral sequence and tell us for what kinds of (brave new) rings THH is known [Bö1, Bö2, S00, EKMM97, Gr, A05, AHL10, McCS93]...

8. **Higher and iterated THH** Topological Hochschild homology of a commutative ring spectrum R can be expressed as $R \otimes \mathbb{S}^1$. Taking an n-torus (or an n-sphere) instead of a 1-sphere gives iterated (or higher) topological Hochschild homology. Higher THH can be used to calculate iterated THH using the cell structure on the n-torus. What is known? How can we compute these things? [BCD10, CDD11, DLR, Ve]

9. **Topological André Quillen (co)homoloogy and Postnikov towers of commutative S-algebras** For connective commutative ring spectra there is a Postnikow tower in the setting of commutative ring spectra refining the ordinary one. The corresponding k-invariants live in topological André-Quillen cohomology. Explain the definition of this cohomology theory and how to build the tower [B99].

10. **Obstruction theory** In algebra we can usually see whether a ring is associative or commutative or not. For homotopy ring spectra it can be hard to see whether the multiplication can be rigidified to a strict ring structure. This can be done by showing that the ordinary Postnikov tower of a spectrum can be refined to one in the world of associative or commutative ring spectra [B99] or obstruction theory can help: [Ro89, Ro03, A11, GH04, GH].

11. **Algebraic K-theory of rings and ring spectra – intro** Describe Quillen's definition of algebraic K-theory of rings [Q72, Q73], and the value of $\pi_n K(R)$ for small n [We13]. What is Waldhausen's $S_\cdot$-construction [DGM13]?

12. **Algebraic K-theory of rings and ring spectra – some big theorems** Algebraic K-theory is notoriously hard to calculate. Anything that helps is welcome! There are several deep results that reduce that calculation of the algebraic K-theory of rings to other algebraic K-groups and these might be known or 'easier' to calculate. Give an overview (see for instance [We13, St89, G87, Schli06, Ta12],[DGM13, I.2.7]) and focus on one result, for instance on the fundamental theorem comparing $K_*(R[t])$ to $K_*(R)$.

13. **Dennis trace map** Introduce the Dennis trace map from algebraic K-theory to Hochschild homology and variants of cyclic homology. What is topological cyclic homology? [BHM93, D04, DGM13, Mad94, BGT14]

14. **Topological cyclic homology and the cyclotomic trace map** Report on some calculations of TC and of the cyclotomic trace map ([Mad94, BöMa94, D], see also [AGHL14] for a recent result). For background on the deRham-Witt complex see [H1, H2]

15. **Algebraic K-theory of ring spectra – geometric features of algebraic K-theory** What should arithmetic properties of ring spectra be? We probably don't know, but we do know that algebraic K-theory of ring spectra sometimes carries geometric information: Waldhausen's A-theory is related to geometric topology, algebraic K-theory of connective complex K-theory is related to a categorified version of vector bundles (2-vector bundles), Stolz and Teichner aim at describing the spectrum of topological modular forms via field theories and K-theory. Give an overview over these examples and discuss one case in more detail ([Wa85, BDRR13, BDRR11, O12, R-ICM, ST04, ST11]).

16. **Algebraic K-theory of ring spectra – localization sequences, logarithmic structures** Blumberg and Mandell [BlM08] proved that there is a localization sequence, *i.e.*, a cofiber sequence

$$K(\mathbb{Z}) \to K(ku) \to K(KU).$$

This result was conjectured by Rognes. Algebraic K-theory of connective ring spectra can be approached via trace methods but for periodic spectra like periodic complex K-theory, KU, this is not the case, so far. Rognes developed the notion of logarithmic ring spectra [R09] and in joint work with Sagave and Schlichtkrull they calculated logarithmic topological Hochschild homology. Conjecturally there is a trace map from $K(KU)$ to log-THH [BlM08, R09, RSS, Sa14]. Logarithmic structures already appear in [HM03].

17. **Rognes' red-shift conjecture** What is the red-shift conjecture? What evidence is there to support this conjecture? What do we know about Galois descent for algebraic K-theory? ([R-ICM, R-MSRI] and references therein, [AR02, introduction].)

18. **Discussion: What are future directions?**

References

[Ad74] Adams, J. F.. Stable homotopy and generalised homology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974. x+373 pp.

[ABGHR] Ando, Matthew; Blumberg, Andrew J.; Gepner, David J.; Hopkins, Michael J.; Rezk, Charles. Units of ring spectra and Thom spectra, preprint arXiv:0810.4535.

[A11] Angeltveit, Vigleik. Uniqueness of Morava K-theory. Compos. Math. 147 (2011), no. 2, 633–648.

[AHL10] Angeltveit, Vigleik; Hill, Michael A.; Lawson, Tyler. Topological Hochschild homology of ℓ and ko. Amer. J. Math. 132 (2010), no. 2, 297–330.

[AGHL14] Angeltveit, Vigleik; Gerhardt, Teena; Hill, Michael A.; Lindenstrauss, Ayelet. On the algebraic K-theory of truncated polynomial algebras in several variables. J. K-Theory 13 (2014), no. 1, 57–81.

[AG14] Antieau, Benjamin; Gepner, David. Brauer groups and étale cohomology in derived algebraic geometry. Geom. Topol. 18 (2014), no. 2, 1149–1244.

[A05] Ausoni, Christian. Topological Hochschild homology of connective complex K-theory. Amer. J. Math. 127 (2005), no. 6, 1261–1313.

[AR02] Ausoni, Christian; Rognes, John. Algebraic K-theory of topological K-theory. Acta Math. 188 (2002), no. 1, 1–39.

[BDRR11] Baas, Nils A.; Dundas, Bjørn Ian; Richter, Birgit; Rognes, John. Stable bundles over rig categories. J. Topol. 4 (2011), no. 3, 623–640.

[BDRR13] Baas, Nils A.; Dundas, Bjørn Ian; Richter, Birgit; Rognes, John. Ring completion of rig categories. J. Reine Angew. Math. 674 (2013), 43–80.

[BR05] Baker, Andrew; Richter, Birgit. Invertible modules for commutative S-algebras with residue fields. Manuscripta Math. 118 (2005), no. 1, 99–119.

[BRS12] Baker, Andrew; Richter, Birgit; Szymik, Markus. Brauer groups for commutative S-algebras. J. Pure Appl. Algebra 216 (2012), no. 11, 2361–2376.

[B99] Basterra, M. André-Quillen cohomology of commutative S-algebras. J. Pure Appl. Algebra 144 (1999), no. 2, 111–143.

[BGT14] Blumberg, Andrew J.; Gepner, David; Tabuada, Gonçalo. Uniqueness of the multiplicative cyclotomic trace. Adv. Math. 260 (2014), 191–232.

[BlM08] Blumberg, Andrew J.; Mandell, Michael A. The localization sequence for the algebraic K-theory of topological K-theory. Acta Math. 200 (2008), no. 2, 155–179.

[Bö1] Bökstedt, Marcel, Topological Hochschild homology, preprint.

[Bö2] Bökstedt, Marcel, The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$, preprint.

[BHM93] Bökstedt, M.; Hsiang, W. C.; Madsen, I. The cyclotomic trace and algebraic K-theory of spaces. Invent. Math. 111 (1993), no. 3, 465–539.

[BöMa94] Bökstedt, M.; Madsen, I. Topological cyclic homology of the integers. K-theory (Strasbourg, 1992). Astérisque No. 226 (1994), 7–8, 57–143.

[BCD10] Brun, Morten; Carlsson, Gunnar; Dundas, Bjrn Ian. Covering homology. Adv. Math. 225 (2010), no. 6, 3166–3213.

[CDD11] Carlsson, Gunnar; Douglas, Cristopher L.; Dundas, Bjørn Ian. Higher topological cyclic homology and the Segal conjecture for tori. Adv. Math. 226 (2011), 1823–1874.

[DS07] Dugger, Daniel; Shipley, Brooke. Topological equivalences for differential graded algebras. Adv. Math. 212 (2007), no. 1, 37–61.

[D97] Dundas, Bjørn Ian, Relative K-theory and topological cyclic homology. Acta Math. 179 (1997), no. 2, 223–242.

[D04] Dundas, Bjørn Ian. The cyclotomic trace for S-algebras. J. London Math. Soc. (2) 70 (2004), no. 3, 659–677. MR2096869 (2005i:19001)

[D] Dundas, Bjørn Ian. The cyclotomic trace preserves operad actions, preprint available at `http://folk.uib.no/nmabd/`

[DGM13] Dundas, Bjørn Ian; Goodwillie, Thomas G.; McCarthy, Randy. The local structure of algebraic K-theory. Algebra and Applications, 18. Springer-Verlag London, Ltd., London, 2013. xvi+435 pp.

[DLR] Dundas, Bjørn Ian, Lindenstrauss, Ayelet; Richter, Birgit. Higher topological Hochschild homology of rings of integers, preprint arXiv:1502.02504.

[EKMM97] Elmendorf, A. D.; Kriz, I.; Mandell, M. A.; May, J. P. Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole. Mathematical Surveys and Monographs, 47. American Mathematical Society, Providence, RI, 1997. xii+249 pp.

[EM04] Elmendorf, A. D.; Mandell, M. A. Permutative categories as a model of connective stable homotopy. Structured ring spectra, 19–31, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[EM09] Elmendorf, A. D.; Mandell, M. A. Permutative categories, multicategories and algebraic K-theory. Algebr. Geom. Topol. 9 (2009), no. 4, 2391–2441.

[GH04] Goerss, P. G.; Hopkins, M. J. Moduli spaces of commutative ring spectra. Structured ring spectra, 151–200, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[GH] Goerss, P. G.; Hopkins, M. J.. Moduli problems for structured ring spectra, preprint, available as `http://www.math.northwestern.edu/~pgoerss/spectra/obstruct.pdf`

[G87] Grayson, Daniel R. Exact sequences in algebraic K-theory. Illinois J. Math. 31 (1987), no. 4, 598–617

[Gr] Greenlees, John P. C.. Ausoni-Bökstedt duality for topological Hochschild homology, preprint arXiv:1406.2162.

[H1] Hesselholt, Lars. Lecture notes on the big de Rham-Witt complex, preprint available at `http://www.math.nagoya-u.ac.jp/~larsh/papers/s04/`

[H2] Hesselholt, Lars. The big de Rham-Witt complex, preprint available at `http://www.math.nagoya-u.ac.jp/~larsh/papers/028/`

[HM03] Hesselholt, Lars; Madsen, Ib. On the K-theory of local fields. Ann. of Math. (2) 158 (2003), no. 1, 1–113.

[HM97] Hesselholt, Lars; Madsen, Ib. On the K-theory of finite algebras over Witt vectors of perfect fields. Topology 36 (1997), no. 1, 29–101.

[HMS94] Hopkins, Michael J.; Mahowald, Mark; Sadofsky, Hal. Constructions of elements in Picard groups. Topology and representation theory (Evanston, IL, 1992), Contemp. Math., 158, Amer. Math. Soc., Providence, RI, 1994, 89–126.

[Ho] Hovey, Mark, Smith ideals of structured ring spectra, preprint arXiv:1401.2850

[HS99] Hovey, Mark; Sadofsky, Hal. Invertible spectra in the $E(n)$-local stable homotopy category. J. London Math. Soc. (2) 60 (1999), no. 1, 284–302.

[HSS00] Hovey, Mark; Shipley, Brooke; Smith, Jeff. Symmetric spectra. J. Amer. Math. Soc. 13 (2000), no. 1, 149–208.

[Ka10] Karamanov, Nasko On Hopkins' Picard group Pic_2 at the prime 3. Algebr. Geom. Topol. 10 (2010), no. 1, 275–292.

[Ko96] Kochman, S. O. Bordism, stable homotopy and Adams spectral sequences. Fields Institute Monographs, 7. American Mathematical Society, Providence, RI, 1996. xiv+272 pp.

[L09] Lawson, Tyler. Commutative Γ-rings do not model all commutative ring spectra. Homology, Homotopy Appl. 11 (2009), no. 2, 189–194.

[Le91] Lewis, L. Gaunce, Jr., *Is there a convenient category of spectra?*, J. Pure Appl. Algebra 73 (1991), no. 3, 233–246.

[Lo98] Loday, Jean-Louis. Cyclic homology. Appendix E by María O. Ronco. Second edition. Chapter 13 by the author in collaboration with Teimuraz Pirashvili. Grundlehren der Mathematischen Wissenschaften 301. Springer-Verlag, Berlin, 1998. xx+513 pp.

[Mad94] Madsen, Ib. Algebraic K-theory and traces. Current developments in mathematics, 1995 (Cambridge, MA), 191–321, Int. Press, Cambridge, MA, 1994.

[M10] Mandell, Michael A. An inverse K-theory functor. Doc. Math. 15 (2010), 765–791. MR2735988 (2012a:19005)

[Ma] Mathew; Akhil. The Galois group of a stable homotopy theory, preprint arXiv:1404.2156.

[MaSt] Mathew, Akhil; Stojanoska, Vesna. The Picard group of topological modular forms via descent theory, preprint arXiv:1409.7702.

[M77] May, J. Peter. E_∞ ring spaces and E_∞ ring spectra. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977. 268 pp.

[M78] May, J. P. The spectra associated to permutative categories. Topology 17 (1978), no. 3, 225–228.

[M09a] May, J. P. The construction of E_∞ ring spaces from bipermutative categories. New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 331–365, Geom. Topol. Monogr., 16, Geom. Topol. Publ., Coventry, 2009.

[M09b] May, J. P. What are E_∞ ring spaces good for? New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 331–365, Geom. Topol. Monogr., 16, Geom. Topol. Publ., Coventry, 2009.

[McCS93] McClure, J. E.; Staffeldt, R. E. On the topological Hochschild homology of bu. I. Amer. J. Math. 115 (1993), no. 1, 1–45.

[Mi] Haynes Miller, Notes on Cobordism, lecture notes available at `http://www-math.mit.edu/~hrm/papers/cobordism.pdf`

[O12] Osorno, Angelica. Spectra associated to symmetric monoidal bicategories. Algebr. Geom. Topol. 12 (2012), no. 1, 307–342. MR2916278

[Q72] Quillen, Daniel. On the cohomology and K-theory of the general linear groups over a finite field. Ann. of Math. (2) 96 (1972), 552–586.

[Q73] Quillen, Daniel. Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[Ra84] Ravenel, Douglas C. Localization with respect to certain periodic homology theories. Amer. J. Math. 106 (1984), no. 2, 351–414. MR0737778 (85k:55009)

[Ra86] Ravenel, Douglas C. Complex cobordism and stable homotopy groups of spheres. Pure and Applied Mathematics, 121. Academic Press, Inc., Orlando, FL, 1986. xx+413 pp.

[Ro89] Robinson, Alan. Obstruction theory and the strict associativity of Morava K-theories. Advances in homotopy theory (Cortona, 1988), 143–152, London Math. Soc. Lecture Note Ser., 139, Cambridge Univ. Press, Cambridge, 1989. MR1055874 (91d:55003)

[Ro03] Robinson, Alan. Gamma homology, Lie representations and E_∞ multiplications. Invent. Math. 152 (2003), no. 2, 331–348. MR1974890 (2004c:55020)

[R08] Rognes, John. Galois extensions of structured ring spectra. Stably dualizable groups. Mem. Amer. Math. Soc. 192 (2008), no. 898, viii+137 pp. MR2387923 (2009c:55007)

[R-ICM] Rognes, John. Algebraic K-theory of strict ring spectra. Contribution to the proceedings of the ICM 2014 in Seoul, arXiv:1403.5998.

[R-MSRI] Rognes, John, Chromatic red shift. Notes for MSRI talk, arXiv:1403.4838.

[R09] Rognes, John. Topological logarithmic structures. New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 401–544, Geom. Topol. Monogr., 16, Geom. Topol. Publ., Coventry, 2009.

[RSS] Rognes, John, Sagave, Steffen, Schlichtkrull, Christian. Localization sequences for logarithmic topological Hochschild homology, preprint arXiv:1402.1317.

[R98] Rudyak, Yuli B.. On Thom spectra, orientability, and cobordism. With a foreword by Haynes Miller. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xii+587 pp.

[Sa14] Sagave, Steffen. Logarithmic structures on topological K-theory spectra. Geom. Topol. 18 (2014), no. 1, 447490.

[Sa] Sagave, Steffen. Spectra of units for periodic ring spectra and group completion of graded E-infinity spaces, preprint arXiv:1111.6731.

[Schli06] Schlichting, Marco. Negative K-theory of derived categories. Math. Z. 253 (2006), no. 1, 97–134.

[Schl09] Schlichtkrull, Christian. Thom spectra that are symmetric spectra. Doc. Math. 14 (2009), 699–748.

[Schw99] Schwede, Stefan. Stable homotopical algebra and Γ-spaces. Math. Proc. Cambridge Philos. Soc. 126 (1999), no. 2, 329–356.

[Schw] Schwede, Stefan, An untitled book project about symmetric spectra, latest version available at `www.math.uni-bonn.de/people/schwede/SymSpec-v3.pdf`

[S74] Segal, Graeme. Categories and cohomology theories. Topology 13 (1974), 293–312. MR0353298 (50 #5782)

[ShSh79] Shimada, Nobuo; Shimakawa, Kazuhisa. Delooping symmetric monoidal categories. Hiroshima Math. J. 9 (1979), no. 3, 627–645.

[S00] Shipley, Brooke. Symmetric spectra and topological Hochschild homology. K-Theory 19 (2000), no. 2, 155–183. MR1740756 (2001h:55010)

[St89] Staffeldt, Ross E. On fundamental theorems of algebraic K-theory. K-Theory 2 (1989), no. 4, 511–532.

[ST04] Stolz, Stephan; Teichner, Peter. What is an elliptic object? Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge (2004), 247–343.

[ST11] Stolz, Stephan; Teichner, Peter. Supersymmetric field theories and generalized cohomology. Mathematical foundations of quantum field theory and perturbative string theory, Proc. Sympos. Pure Math., 83, Amer. Math. Soc., Providence, RI (2011), 279–340.

[St92] Strickland, N. P.. On the p-adic interpolation of stable homotopy groups. Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990), 4554, London Math. Soc. Lecture Note Ser., 176, Cambridge Univ. Press, Cambridge, 1992.

[Ta12] Tabuada, Gonçalo. The fundamental theorem via derived Morita invariance, localization, and $\mathbb{A}^1$-homotopy invariance. J. K-Theory 9 (2012), no. 3, 407–420

[T95] Thomason, R. W. Symmetric monoidal categories model all connective spectra. Theory Appl. Categ. 1 (1995), No. 5, 78–118.

[Ve] Veen, Torleif. Detecting Periodic Elements in Higher Topological Hochschild Homology, preprint arXiv:1312.5699.

[Wa85] Waldhausen, Friedhelm. Algebraic K-theory of spaces. Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., 1126, Springer, Berlin, 1985, 318–419.

[We13] Weibel, Charles A. The K-book. An introduction to algebraic K-theory. Graduate Studies in Mathematics, 145. American Mathematical Society, Providence, RI, 2013. xii+618 pp.