Solution exercise 1, week 6

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Suppose that $[-,(Y,y_0)]_*:\mathbf{Top}_*\to\mathbf{Gr}$. Let $\mu : Y\times Y\to Y$ be a based continuous map such that $[\mu]_*=[pr_1]_*\cdot[pr_2]_*$. Prove that

$$\mu\circ(\mu\times\ Id_Y)\simeq_*\mu\circ(Id_Y\times\mu).$$

Proof. We want to show that $[\mu\circ(\mu\times Id_Y)]_*=[\mu\circ(Id_Y\times\mu)]_*$, i.e. that the following diagram commutes "homotopically".

$$\begin{array}{ccc} Y\times Y\times Y & \xrightarrow{\mu\times Id_Y} & Y\times Y \\ {\scriptstyle Id_Y\times\mu}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ Y\times Y & \xrightarrow[\mu]{} & Y \end{array}$$

Since the mapping $(\mu\times Id_Y)\in\mathbf{Top}_*(Y\times Y\times Y,Y\times Y)$, under the hypothesis that the functor $[-,(Y,y_0)]_*$ factorizes through the category of groups $\mathbf{Gr}$, and by definition of a functor, we conclude that

$$\begin{array}{rccc} (\mu\times Id_Y)^*: & [Y\times Y,Y]_* & \longrightarrow & [Y\times Y\times Y,Y] \\ & [f]_* & \longmapsto & [f\circ(\mu\times Id_Y)]_* \end{array}$$

is a homomorphism of groups. Therefore,

$$\begin{aligned}[\mu\circ(\mu\times Id_Y)]_* &= (\mu\times Id_Y)^*([\mu]_*)=(\mu\times Id_Y)^*([pr_1]_*\cdot[pr_2]_*)\\ &=(\mu\times Id_Y)^*([pr_1]_*)\cdot(\mu\times Id_Y)^*([pr_2]_*)\\ &=[pr_1\circ(\mu\times Id_Y)]_*\cdot[pr_2\circ(\mu\times Id_Y)]_*.\end{aligned}$$

For all $(y_1,y_2,y_3)\in Y\times Y\times Y$, we have

$$pr_2\circ(\mu\times Id_Y)(y_1,y_2,y_3)=pr_2(\mu(y_1,y_2),y_3)=y_3=pr_3(y_1,y_2,y_3),$$

thus, $[pr_2\circ(\mu\times Id_Y)]_*=[pr_3]_*$.

Now, let us define the mapping

$$\begin{array}{rccc} pr_{1,2}: & Y\times Y\times Y & \longrightarrow & Y\times Y \\ & (y_1,y_2,y_3) & \longmapsto & (y_1,y_2), \end{array}$$

which is clearly continuous (by universal property of the product topology).

By the same reasoning applied to $(\mu \times Id_Y)^*$, we conclude that $pr_{1,2}^*$ is also a homomorphism of groups. Moreover, for all $(y_1, y_2, y_3) \in Y \times Y \times Y$, we have

$$pr_1 \circ (\mu \times Id_Y)(y_1, y_2, y_3) = pr_1(\mu(y_1, y_2), y_3) = \mu(y_1, y_2) = \mu \circ pr_{1,2}(y_1, y_2, y_3),$$

therefore,

$$\begin{aligned}
[pr_1 \circ (\mu \times Id_Y)]_* &= [\mu \circ pr_{1,2}]_* = pr_{1,2}^*([\mu]_*) = pr_{1,2}^*([pr_1]_* \cdot [pr_2]_*) \\
&= pr_{1,2}^*([pr_1]_*) \cdot pr_{1,2}^*([pr_2]_*) = [pr_1 \circ pr_{1,2}]_* \cdot [pr_2 \circ pr_{1,2}]_* \\
&= [pr_1]_* \cdot [pr_2]_*.
\end{aligned}$$

We conclude that

$$[\mu \circ (\mu \times Id_Y)]_* = [pr_1]_* \cdot [pr_2]_* \cdot [pr_3]_*.$$

With a very similar argument we can see that $[\mu \circ (Id_Y \times \mu)]_*$ gives the same result. Explicitly, we have that $(Id_Y \times \mu)^*$ is a homomorphism of groups, therefore,

$$\begin{aligned}
[\mu \circ (Id_Y \times \mu)]_* &= (Id_Y \times \mu)^*([\mu]_*) = (Id_Y \times \mu)^*([pr_1]_* \cdot [pr_2]_*) \\
&= (Id_Y \times \mu)^*([pr_1]_*) \cdot (Id_Y \times \mu)^*([pr_2]_*) \\
&= [pr_1 \circ (Id_Y \times \mu)]_* \cdot [pr_2 \circ (Id_Y \times \mu)]_*.
\end{aligned}$$

For all $(y_1, y_2, y_3) \in Y \times Y \times Y$, we have

$$pr_1 \circ (Id_Y \times \mu)(y_1, y_2, y_3) = pr_1(y_1, \mu(y_2, y_3)) = y_1 = pr_1(y_1, y_2, y_3),$$

thus, $[pr_1 \circ (Id_Y \times \mu)]_* = [pr_1]_*$. We define the continuous mapping $pr_{2,3}$ such that $pr_{2,3}(y_1, y_2, y_3) = (y_2, y_3)$ and obtain

$$pr_2 \circ (Id_Y \times \mu)(y_1, y_2, y_3) = pr_2(y_1, \mu(y_2, y_3)) = \mu(y_2, y_3) = \mu \circ pr_{2,3}(y_1, y_2, y_3).$$

Therefore, since $pr_{2,3}^*$ is a homomorphism of groups (same reasoning as before),

$$\begin{aligned}
[pr_2 \circ (Id_Y \times \mu)]_* &= [\mu \circ pr_{2,3}]_* = pr_{2,3}^*([\mu]_*) = pr_{2,3}^*([pr_1]_* \cdot [pr_2]_*) \\
&= pr_{2,3}^*([pr_1]_*) \cdot pr_{2,3}^*([pr_2]_*) = [pr_1 \circ pr_{2,3}]_* \cdot [pr_2 \circ pr_{2,3}]_* \\
&= [pr_2]_* \cdot [pr_3]_*.
\end{aligned}$$

Hence,

$$[\mu \circ (Id_Y \times \mu)]_* = [pr_1]_* \cdot [pr_2]_* \cdot [pr_3]_*.$$

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