p-local finite groups and partial groups

Oihana Garaialde

Young Topologists' Meeting 2015,
EPFL, Ecublens, Switzerland

eman ta zabal zazu
Universidad del País Vasco
Euskal Herriko Unibertsitatea

UNIVERSIDAD DE MÁLAGA

Fusion Systems

Let G be a finite group and let p be a prime dividing the order of G.

Fusion Systems

Let G be a finite group and let p be a prime dividing the order of G. A fusion system $\mathcal{F}_S(G)$ is a category that will decrypt the p-structure of G.

Formally,

Fusion Systems

Let G be a finite group and let p be a prime dividing the order of G. A fusion system $\mathcal{F}_S(G)$ is a category that will decrypt the p-structure of G.

Formally, let S be a Sylow p-subgroup of G. Then, the category $\mathcal{F}_S(G)$ takes as objects,

$Ob(\mathcal{F}_S(G))$, the subgroups P of S,

Fusion Systems

Let G be a finite group and let p be a prime dividing the order of G. A fusion system $\mathcal{F}_S(G)$ is a category that will decrypt the p-structure of G.

Formally, let S be a Sylow p-subgroup of G. Then, the category $\mathcal{F}_S(G)$ takes as objects,

$Ob(\mathcal{F}_S(G))$, the subgroups P of S,

and as morphisms,

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,S) = \mathrm{Hom}_G(P,S) = \{\psi : P \to S | \psi = c_g \text{ for } g \in G\}.$$

Let G be a finite group and let p be a prime dividing the order of G. A fusion system $\mathcal{F}_S(G)$ is a category that will decrypt the p-structure of G.
Formally, let S be a Sylow p-subgroup of G. Then, the category $\mathcal{F}_S(G)$ takes as objects,
$Ob(\mathcal{F}_S(G))$, the subgroups P of S,
and as morphisms,

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, S) = \mathrm{Hom}_G(P, S) = \{\psi : P \to S | \psi = c_g \text{ for } g \in G\}.$$

More generally, one can define a fusion system $\mathcal{F}$ over any finite p-group S.

Fusion Systems

Let G be a finite group and let p be a prime dividing the order of G. A fusion system $\mathcal{F}_S(G)$ is a category that will decrypt the p-structure of G.

Formally, let S be a Sylow p-subgroup of G. Then, the category $\mathcal{F}_S(G)$ takes as objects,

$Ob(\mathcal{F}_S(G))$, the subgroups P of S,

and as morphisms,

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, S) = \mathrm{Hom}_G(P, S) = \{\psi : P \to S | \psi = c_g \text{ for } g \in G\}.$$

More generally, one can define a fusion system $\mathcal{F}$ over any finite p-group S. In that case, one introduces a notion of *saturated* fusion system.

Let G be a finite group and let p be a prime dividing the order of G. A fusion system $\mathcal{F}_S(G)$ is a category that will decrypt the p-structure of G.
Formally, let S be a Sylow p-subgroup of G. Then, the category $\mathcal{F}_S(G)$ takes as objects,
$Ob(\mathcal{F}_S(G))$, the subgroups P of S,
and as morphisms,

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,S) = \mathrm{Hom}_G(P,S) = \{\psi : P \to S | \psi = c_g \text{ for } g \in G\}.$$

More generally, one can define a fusion system $\mathcal{F}$ over any finite p-group S. In that case, one introduces a notion of *saturated* fusion system.
A fusion system is saturated if it satisfies two axioms (*Axiom of Sylow* and *Extension axiom*) that mimic the previous case.

Remarks:

Remarks:

1) In the previous cases, we say that the fusion system is *finite*.

Fusions Systems

Remarks:

1) In the previous cases, we say that the fusion system is *finite*.
2) Given a p-group S, there can be several different fusion systems over S.

Remarks:

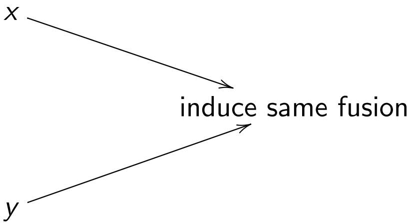
1) In the previous cases, we say that the fusion system is *finite*.
2) Given a p-group S, there can be several different fusion systems over S.
3) There is a notion of *compact* fusion system (that generalizes Lie compact groups): a compact fusion system is a fusion system over a discrete p-toral group P. Here, P is an extension of a discrete p-torus $P_0 \cong (\mathbb{Z}/p^\infty)^r$ by a finite p-group.

Fusions Systems

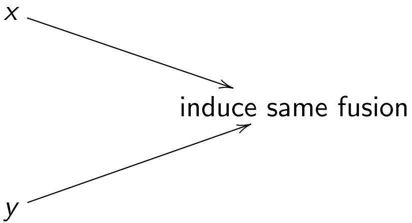
Remarks:

1) In the previous cases, we say that the fusion system is *finite*.
2) Given a p-group S, there can be several different fusion systems over S.
3) There is a notion of *compact* fusion system (that generalizes Lie compact groups): a compact fusion system is a fusion system over a discrete p-toral group P. Here, P is an extension of a discrete p-torus $P_0 \cong (\mathbb{Z}/p^\infty)^r$ by a finite p-group.
4) The finite fusion system is a particular case of a compact fusion system with $r = 0$.

Centric linking system

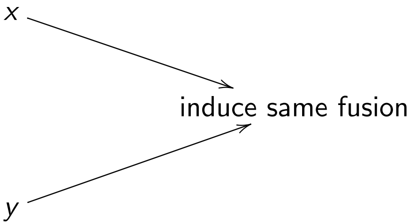


Centric linking system



E.g., all the central elements induce identity homomorphisms.

Centric linking system



E.g., all the central elements induce identity homomorphisms.

Centric linking systems aim to tell apart such elements.

More formally, a centric linking system of G over S, $\mathcal{L}_S(G)$, is a category that takes as objects

Centric Linking systems

More formally, a centric linking system of G over S, $\mathcal{L}_S(G)$, is a category that takes as objects
the $\mathcal{F}_S(G)$-centric objects, that is, the subgroups P of S that satisfy $C_G(P) = Z(P) \times \{\text{coprime to } p \text{ part}\}$. This coprime to p part is a subgroup of $C_G(P)$.

Centric Linking systems

More formally, a centric linking system of G over S, $\mathcal{L}_S(G)$, is a category that takes as objects
the $\mathcal{F}_S(G)$-centric objects, that is, the subgroups P of S that satisfy $C_G(P) = Z(P) \times \{\text{coprime to } p \text{ part}\}$. This coprime to p part is a subgroup of $C_G(P)$.
And this category takes as morphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{L}_S(G)}(P, S) &= N_G(P, S)/\{\text{coprime to } p \text{ part}\}\\ &= \{g \in G |\, g^{-1}Pg \leq S\}/\{\text{coprime to } p \text{ part}\}.\end{aligned}$$

Remark

Centric linking systems can also be defined (abstractly) over a p-group, denoted by $\mathcal{L}$ (as fusion systems $\mathcal{F}$).

p-local finite groups

Remark

Centric linking systems can also be defined (abstractly) over a p-group, denoted by $\mathcal{L}$ (as fusion systems $\mathcal{F}$).

Definition

Let p be a prime number. A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p-group, $\mathcal{F}$ is a (saturated) fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.

p-local finite groups

Remark

Centric linking systems can also be defined (abstractly) over a p-group, denoted by $\mathcal{L}$ (as fusion systems $\mathcal{F}$).

Definition

Let p be a prime number. A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p-group, $\mathcal{F}$ is a (saturated) fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.

Example

$(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$ where $S \in Syl_p(G)$ for some prime $p||G|$.

p-local finite groups

There is a relation between the p-local structure of a group G (encoded by algebraic categories such as $\mathcal{F}_S(G)$ and $\mathcal{L}_S(G)$) and the homotopy type of its classifying space $BG^\wedge_p$.

There is a relation between the p-local structure of a group G (encoded by algebraic categories such as $\mathcal{F}_S(G)$ and $\mathcal{L}_S(G)$) and the homotopy type of its classifying space $BG_p^\wedge$.
In fact, it is known that

p-local finite groups

There is a relation between the p-local structure of a group G (encoded by algebraic categories such as $\mathcal{F}_S(G)$ and $\mathcal{L}_S(G)$) and the homotopy type of its classifying space $BG^\wedge_p$.

In fact, it is known that

1. $|\mathcal{L}_S(G)|^\wedge_p \simeq BG^\wedge_p$ where $|\cdot|^\wedge_p$ denotes the p-completed nerve of a category.

p-local finite groups

There is a relation between the p-local structure of a group G (encoded by algebraic categories such as $\mathcal{F}_S(G)$ and $\mathcal{L}_S(G)$) and the homotopy type of its classifying space $BG^{\wedge}_p$.

In fact, it is known that

1. $|\mathcal{L}_S(G)|^{\wedge}_p \simeq BG^{\wedge}_p$ where $|\cdot|^{\wedge}_p$ denotes the p-completed nerve of a category.
2. $BG^{\wedge}_p \simeq BG'^{\wedge}_p$ if and only if $\mathcal{F}_G(S) \cong \mathcal{F}'_G(S')$ where S and S' are p-Sylow subgroups of G and G', respectively.

There is a relation between the p-local structure of a group G (encoded by algebraic categories such as $\mathcal{F}_S(G)$ and $\mathcal{L}_S(G)$) and the homotopy type of its classifying space $BG_p^\wedge$.

In fact, it is known that

1. $|\mathcal{L}_S(G)|_p^\wedge \simeq BG_p^\wedge$ where $|\cdot|_p^\wedge$ denotes the p-completed nerve of a category.
2. $BG_p^\wedge \simeq BG_p'^\wedge$ if and only if $\mathcal{F}_G(S) \cong \mathcal{F}'_G(S')$ where S and S' are p-Sylow subgroups of G and G', respectively.

Definition

If there is no a finite group G such that

$$(S, \mathcal{F}_S(G), \mathcal{L}_S(G)) = (S, \mathcal{F}, \mathcal{L})$$

with S a p-Sylow subgroup of G, then we say that $(S, \mathcal{F}, \mathcal{L})$ is an *exotic* p-local finite group.

Remark

There are p-local finite groups that do not arise from a finite group. So, p-local finite groups are considered as a generalization of finite groups.

Remark

There are p-local finite groups that do not arise from a finite group. So, p-local finite groups are considered as a generalization of finite groups.
Notice that if $(S, \mathcal{F}, \mathcal{L})$ is an exotic p-local finite group, then $|\mathcal{L}|_p^\wedge$ is not homotopic equivalent to a classigying space of a finite group. However, it "behaves" as if it was.

Remark

There are p-local finite groups that do not arise from a finite group. So, p-local finite groups are considered as a generalization of finite groups.
Notice that if $(S, \mathcal{F}, \mathcal{L})$ is an exotic p-local finite group, then $|\mathcal{L}|_p^\wedge$ is not homotopic equivalent to a classigying space of a finite group. However, it "behaves" as if it was.

We have that:

p-local finite groups

Remark

There are p-local finite groups that do not arise from a finite group. So, p-local finite groups are considered as a generalization of finite groups.
Notice that if $(S, \mathcal{F}, \mathcal{L})$ is an exotic p-local finite group, then $|\mathcal{L}|_p^\wedge$ is not homotopic equivalent to a classigying space of a finite group. However, it "behaves" as if it was.

We have that:

$$\text{Given a finite group } G \Rightarrow \text{can construct } \mathcal{F}_G(S)$$
$$\Rightarrow \text{can construct } \mathcal{L} \text{ with } |\mathcal{L}|_p^\wedge \simeq BG_p^\wedge.$$

Remark

There are p-local finite groups that do not arise from a finite group. So, p-local finite groups are considered as a generalization of finite groups.
Notice that if $(S, \mathcal{F}, \mathcal{L})$ is an exotic p-local finite group, then $|\mathcal{L}|_p^\wedge$ is not homotopic equivalent to a classgiying space of a finite group. However, it "behaves" as if it was.

We have that:

$$\text{Given a finite group } G \Rightarrow \text{can construct } \mathcal{F}_G(S)$$
$$\Rightarrow \text{can construct } \mathcal{L} \text{ with } |\mathcal{L}|_p^\wedge \simeq BG_p^\wedge.$$

Question: For each $\mathcal{F}$, is there an associated $\mathcal{L}$?

p-local finite groups

Remark

There are p-local finite groups that do not arise from a finite group. So, p-local finite groups are considered as a generalization of finite groups.
Notice that if $(S, \mathcal{F}, \mathcal{L})$ is an exotic p-local finite group, then $|\mathcal{L}|_p^\wedge$ is not homotopic equivalent to a classgigying space of a finite group. However, it "behaves" as if it was.

We have that:

$$\text{Given a finite group } G \Rightarrow \text{can construct } \mathcal{F}_G(S)$$
$$\Rightarrow \text{can construct } \mathcal{L} \text{ with } |\mathcal{L}|_p^\wedge \simeq BG_p^\wedge.$$

Question: For each $\mathcal{F}$, is there an associated $\mathcal{L}$? Is it unique?

Motivation: Partial groups

Partial groups (and *localities*) were introduced to answer this question.

Motivation: Partial groups

Partial groups (and *localities*) were introduced to answer this question. In fact, Andy Chermak showed that for each fusion system $\mathcal{F}$, there exists a unique centric linking system $\mathcal{L}$ associated to $\mathcal{F}$.

Motivation: Partial groups

Partial groups (and *localities*) were introduced to answer this question. In fact, Andy Chermak showed that for each fusion system $\mathcal{F}$, there exists a unique centric linking system $\mathcal{L}$ associated to $\mathcal{F}$. (Using the idea of A. Chermak, B. Oliver showed the same result based on the computation of higher derived functors of certain inverse limits).

Partial Groups

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.
   Set $\Pi(\emptyset) = 1$. A mapping $\Pi : \mathbb{D} \to \mathcal{L}$ is a *product* if:

Partial Groups

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.
   Set $\Pi(\emptyset) = 1$. A mapping $\Pi : \mathbb{D} \to \mathcal{L}$ is a *product* if:
2. $\Pi_{|\mathcal{L}}$ is the identity map,

Partial Groups

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.
   Set $\Pi(\emptyset) = 1$. A mapping $\Pi : \mathbb{D} \to \mathcal{L}$ is a *product* if:
2. $\Pi_{|\mathcal{L}}$ is the identity map,
3. if $u \circ v \circ w \in \mathbb{D}$, then $u \circ \Pi(v) \circ w \in \mathbb{D}$ and
   $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$,

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.
   Set $\Pi(\emptyset) = 1$. A mapping $\Pi : \mathbb{D} \to \mathcal{L}$ is a *product* if:
2. $\Pi_{|\mathcal{L}}$ is the identity map,
3. if $u \circ v \circ w \in \mathbb{D}$, then $u \circ \Pi(v) \circ w \in \mathbb{D}$ and
   $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$,
   An *inversion* on $\mathcal{L}$ consists of an involutory bijection $f \to f^{-1}$ on $\mathcal{L}$, together with the mapping $u \to u^{-1}$ on $\mathbb{W}$ given by

$$(x_1, \dots, x_n) \to (x_n^{-1}, \dots, x_1^{-1}).$$

Partial Groups

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.
   Set $\Pi(\emptyset) = 1$. A mapping $\Pi : \mathbb{D} \to \mathcal{L}$ is a *product* if:
2. $\Pi_{|\mathcal{L}}$ is the identity map,
3. if $u \circ v \circ w \in \mathbb{D}$, then $u \circ \Pi(v) \circ w \in \mathbb{D}$ and
   $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$,
   An *inversion* on $\mathcal{L}$ consists of an involutory bijection $f \to f^{-1}$ on $\mathcal{L}$, together with the mapping $u \to u^{-1}$ on $\mathbb{W}$ given by

$$(x_1, \ldots, x_n) \to (x_n^{-1}, \ldots, x_1^{-1}).$$

   We say that $(\mathcal{L}, \mathbb{D}(\mathcal{L}), \Pi, (\cdot)^{-1})$ is a *partial group* if

Definition

Let $\mathcal{L}$ be a non-empty set, let $\mathbb{W} = \mathbb{W}(\mathcal{L})$ be the free monoid in $\mathcal{L}$ and let $\mathbb{D} = \mathbb{D}(\mathcal{L}) \subseteq \mathbb{W}$ be a subset such that

1. $\mathcal{L} \subseteq \mathbb{D}$ and if $u \circ v \in \mathbb{D}$ then, $u, v \in \mathbb{D}$.
   Set $\Pi(\emptyset) = 1$. A mapping $\Pi : \mathbb{D} \to \mathcal{L}$ is a *product* if:
2. $\Pi_{|\mathcal{L}}$ is the identity map,
3. if $u \circ v \circ w \in \mathbb{D}$, then $u \circ \Pi(v) \circ w \in \mathbb{D}$ and
   $\Pi(u \circ v \circ w) = \Pi(u \circ \Pi(v) \circ w)$,
   An *inversion* on $\mathcal{L}$ consists of an involutory bijection $f \to f^{-1}$ on $\mathcal{L}$, together with the mapping $u \to u^{-1}$ on $\mathbb{W}$ given by

   $$(x_1, \ldots, x_n) \to (x_n^{-1}, \ldots, x_1^{-1}).$$

   We say that $(\mathcal{L}, \mathbb{D}(\mathcal{L}), \Pi, (\cdot)^{-1})$ is a *partial group* if
4. if $u \in \mathbb{D}$, then $u^{-1} \circ u \in \mathbb{D}$ and $\Pi(u^{-1} \circ u) = \Pi(\emptyset) = 1$.

Remark

1. *(1) and (4) in the above definition imply that for all* $u \in \mathbb{D}$*, then* $u^{-1} \in \mathbb{D}$.

Remark

1. *(1) and (4) in the above definition imply that for all* $u \in \mathbb{D}$*, then* $u^{-1} \in \mathbb{D}$*.*
2. *If* $\mathbb{D} = \mathbb{W}$*, then* $\mathcal{L}$ *is an (actual) group.*

Remark

1. *(1) and (4) in the above definition imply that for all* $u \in \mathbb{D}$*, then* $u^{-1} \in \mathbb{D}$.
2. *If* $\mathbb{D} = \mathbb{W}$*, then* $\mathcal{L}$ *is an (actual) group.*
3. $\mathbb{D}$*-associativity and* $\mathbb{D}$*-multiplicativity hold:*

Remark

1. *(1) and (4) in the above definition imply that for all* $u \in \mathbb{D}$*, then* $u^{-1} \in \mathbb{D}$*.*
2. *If* $\mathbb{D} = \mathbb{W}$*, then* $\mathcal{L}$ *is an (actual) group.*
3. $\mathbb{D}$*-associativity and* $\mathbb{D}$*-multiplicativity hold:*
   (a) *if* $u \circ v \in \mathbb{D}$*, then*

$$(\Pi(u), \Pi(v)) \in \mathbb{D} \text{ and } \Pi(u \circ v) = \Pi(\Pi(u), \Pi(v)),$$

Remark

1. *(1) and (4) in the above definition imply that for all* $u \in \mathbb{D}$, *then* $u^{-1} \in \mathbb{D}$.
2. *If* $\mathbb{D} = \mathbb{W}$, *then* $\mathcal{L}$ *is an (actual) group.*
3. $\mathbb{D}$*-associativity and* $\mathbb{D}$*-multiplicativity hold:*
   (a) *if* $u \circ v \in \mathbb{D}$, *then*
   $$(\Pi(u), \Pi(v)) \in \mathbb{D} \text{ and } \Pi(u \circ v) = \Pi(\Pi(u), \Pi(v)),$$
   (b) *if* $u \circ v \circ w \in \mathbb{D}$, *then*
   $$\Pi(u \circ v) \circ \Pi(w) = \Pi(u) \circ \Pi(v \circ w).$$

Definition

Let $\mathcal{L}$ be a partial goroup and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then, $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$, $\mathcal{H} \leq \mathcal{L}$, if

Definition

Let $\mathcal{L}$ be a partial goroup and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then, $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$, $\mathcal{H} \leq \mathcal{L}$, if

(a) $\mathcal{H}$ is closed under inversion, that is, $g \in \mathcal{H}$ implies $g^{-1} \in \mathcal{H}$,

Definition

Let $\mathcal{L}$ be a partial goroup and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then, $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$, $\mathcal{H} \leq \mathcal{L}$, if

(a) $\mathcal{H}$ is closed under inversion, that is, $g \in \mathcal{H}$ implies $g^{-1} \in \mathcal{H}$,

(b) $\mathcal{H}$ is closed with respect to products, that is, $\Pi(w) \in \mathcal{H}$ whenever $w \in \mathbb{W}(\mathcal{H}) \cap \mathbb{D}(\mathcal{L})$.

Partial Groups

Definition

Let $\mathcal{L}$ be a partial gorоup and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then, $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$, $\mathcal{H} \leq \mathcal{L}$, if

(a) $\mathcal{H}$ is closed under inversion, that is, $g \in \mathcal{H}$ implies $g^{-1} \in \mathcal{H}$,

(b) $\mathcal{H}$ is closed with respect to products, that is, $\Pi(w) \in \mathcal{H}$ whenever $w \in \mathbb{W}(\mathcal{H}) \cap \mathbb{D}(\mathcal{L})$.

Definition (Conjugation)

Let $\mathcal{L}$ be a partial group and let f be an element in $\mathcal{L}$, we write

Definition

Let $\mathcal{L}$ be a partial gorоup and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then, $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$, $\mathcal{H} \leq \mathcal{L}$, if

(a) $\mathcal{H}$ is closed under inversion, that is, $g \in \mathcal{H}$ implies $g^{-1} \in \mathcal{H}$,

(b) $\mathcal{H}$ is closed with respect to products, that is, $\Pi(w) \in \mathcal{H}$ whenever $w \in \mathbb{W}(\mathcal{H}) \cap \mathbb{D}(\mathcal{L})$.

Definition (Conjugation)

Let $\mathcal{L}$ be a partial group and let f be an element in $\mathcal{L}$, we write

$$\mathbb{D}(f) = \{x \in \mathcal{L} | \ (f^{-1}, x, f) \in \mathbb{D}\}.$$

Definition

Let $\mathcal{L}$ be a partial goroup and let $\mathcal{H}$ be a non-empty subset of $\mathcal{L}$. Then, $\mathcal{H}$ is a *partial subgroup* of $\mathcal{L}$, $\mathcal{H} \leq \mathcal{L}$, if

(a) $\mathcal{H}$ is closed under inversion, that is, $g \in \mathcal{H}$ implies $g^{-1} \in \mathcal{H}$,

(b) $\mathcal{H}$ is closed with respect to products, that is, $\Pi(w) \in \mathcal{H}$ whenever $w \in \mathbb{W}(\mathcal{H}) \cap \mathbb{D}(\mathcal{L})$.

Definition (Conjugation)

Let $\mathcal{L}$ be a partial group and let f be an element in $\mathcal{L}$, we write

$$\mathbb{D}(f) = \{x \in \mathcal{L} | \ (f^{-1}, x, f) \in \mathbb{D}\}.$$

Then, there is a map $c_f : \mathbb{D}(f) \to \mathcal{L}$ called *conjugation map* that sends each element $x \in \mathbb{D}(f)$ to $\Pi(f^{-1}, x, f) = f^{-1}xf \in \mathcal{L}$.

Definition

Let $\mathcal{N}$ be a partial subgroup of $\mathcal{L}$. Then, $\mathcal{N}$ is a *partial normal subgroup* of $\mathcal{L}$, $\mathcal{N} \trianglelefteq \mathcal{L}$,if for all $g \in \mathcal{L}$ and $x \in \mathbb{D}(g) \cap \mathcal{N}$, then $\Pi(g^{-1}, x, g) \in \mathcal{N}$.

Definition

Let $\mathcal{N}$ be a partial subgroup of $\mathcal{L}$. Then, $\mathcal{N}$ is a *partial normal subgroup* of $\mathcal{L}$, $\mathcal{N} \trianglelefteq \mathcal{L}$,if for all $g \in \mathcal{L}$ and $x \in \mathbb{D}(g) \cap \mathcal{N}$, then $\Pi(g^{-1}, x, g) \in \mathcal{N}$.

Definition

Let $\mathcal{L}$ and $\mathcal{L}'$ be two partial groups, let $\beta : \mathcal{L} \to \mathcal{L}'$ be a mapping and let $\beta^* : \mathbb{W}(\mathcal{L}) \to \mathbb{W}(\mathcal{L}')$ be its extension to the free monoids. Then, β is a homomorphism of partial groups if

(a) $\beta^*(\mathbb{D}(\mathcal{L})) \subset \mathbb{D}(\mathcal{L}')$,

(b) $\beta(\Pi(w)) = \Pi(\beta^*(w))$ for all words $w \in \mathbb{D}(\mathcal{L})$.

Example

Let $\mathcal{L}$ be a three element set $\{1, a, b\}$ and let $\mathbb{D}(\mathcal{L})$ be the subset of $\mathbb{W}(\mathcal{L})$ consisting of all words that are obtained from words in $\mathbb{W}(\mathcal{L})$ by deleting all entries equal to 1 and that are alternating string of a's and b's of even or odd length starting with a or b.

Example

Let $\mathcal{L}$ be a three element set $\{1, a, b\}$ and let $\mathbb{D}(\mathcal{L})$ be the subset of $\mathbb{W}(\mathcal{L})$ consisting of all words that are obtained from words in $\mathbb{W}(\mathcal{L})$ by deleting all entries equal to 1 and that are alternating string of a's and b's of even or odd length starting with a or b. Define $\Pi : D(\mathcal{L}) \to \mathcal{L}$ by the formula

$$\Pi(w) = \begin{cases} 1 & \text{if the } a\text{-entries equals to } b\text{-entries in } w, \\ \text{a} & \text{if } a\text{-entries exceeds the number of } b\text{'s in } w, \\ \text{b} & \text{if } b\text{-entries exceeds the number of } a\text{'s in } w. \end{cases}$$

Example

Let $\mathcal{L}$ be a three element set $\{1, a, b\}$ and let $\mathbb{D}(\mathcal{L})$ be the subset of $\mathbb{W}(\mathcal{L})$ consisting of all words that are obtained from words in $\mathbb{W}(\mathcal{L})$ by deleting all entries equal to 1 and that are alternating string of a's and b's of even or odd length starting with a or b. Define $\Pi : D(\mathcal{L}) \to \mathcal{L}$ by the formula

$$\Pi(w) = \begin{cases} 1 & \text{if the } a\text{-entries equals to } b\text{-entries in } w, \\ \text{a} & \text{if } a\text{-entries exceeds the number of } b\text{'s in } w, \\ \text{b} & \text{if } b\text{-entries exceeds the number of } a\text{'s in } w. \end{cases}$$

Define inversion on $\mathcal{L}$ by $1^{-1} = 1$, $a^{-1} = b$ and $b^{-1} = a$. Then, $(\mathcal{L}, \mathbb{D}(\mathcal{L}), \Pi, (\cdot)^{-1})$ is a partial group.

Partial Groups

Example

Let $\mathcal{L}$ be a three element set $\{1, a, b\}$ and let $\mathbb{D}(\mathcal{L})$ be the subset of $\mathbb{W}(\mathcal{L})$ consisting of all words that are obtained from words in $\mathbb{W}(\mathcal{L})$ by deleting all entries equal to 1 and that are alternating string of a's and b's of even or odd length starting with a or b.
Define $\Pi : D(\mathcal{L}) \to \mathcal{L}$ by the formula

$$\Pi(w) = \begin{cases} 1 & \text{if the } a\text{-entries equals to } b\text{-entries in } w, \\ \text{a} & \text{if } a\text{-entries exceeds the number of } b\text{'s in } w, \\ \text{b} & \text{if } b\text{-entries exceeds the number of } a\text{'s in } w. \end{cases}$$

Define inversion on $\mathcal{L}$ by $1^{-1} = 1$, $a^{-1} = b$ and $b^{-1} = a$. Then, $(\mathcal{L}, \mathbb{D}(\mathcal{L}), \Pi, (\cdot)^{-1})$ is a partial group.
We can define a homomorphism of partial groups $\beta : \mathcal{L} \to \mathbb{Z}$ given by $1 \to 0$, $a \to 1$ and $b \to -1$.

Definition

Let p be a prime. Let $\mathcal{L}$ be a partial group, let S be maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$.

Definition

Let p be a prime. Let $\mathcal{L}$ be a partial group, let S be maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$. Let Δ be a collection of subgroups of S that is

Definition

Let p be a prime. Let $\mathcal{L}$ be a partial group, let S be maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$. Let Δ be a collection of subgroups of S that is

a) closed for overgroups,

Definition

Let p be a prime. Let $\mathcal{L}$ be a partial group, let S be maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$. Let Δ be a collection of subgroups of S that is

a) closed for overgroups,

b) closed for conjugation homomorphisms.

Localities

Definition

Let p be a prime. Let $\mathcal{L}$ be a partial group, let S be maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$. Let Δ be a collection of subgroups of S that is

a) closed for overgroups,
b) closed for conjugation homomorphisms.

A word $w = (g_1, \ldots, g_n)$ is in the domain $\mathbb{D}(\mathcal{L})$ if and only if there exist $X_0, \ldots, X_n \in \mathbb{W}(\Delta)$ with $X_{i-1}^{g_i} = X_i$ for all i.

Localities

Definition

Let p be a prime. Let $\mathcal{L}$ be a partial group, let S be maximal in the poset (ordered by inclusion) of finite p-subgroups of $\mathcal{L}$. Let Δ be a collection of subgroups of S that is

a) closed for overgroups,
b) closed for conjugation homomorphisms.

A word $w = (g_1, \ldots, g_n)$ is in the domain $\mathbb{D}(\mathcal{L})$ if and only if there exist $X_0, \ldots, X_n \in \mathbb{W}(\Delta)$ with $X_{i-1}^{g_i} = X_i$ for all i.
Then, we say that $(\mathcal{L}, \Delta, S)$ is a *locality*.

Example

1) Let G be a finite group and S its Sylow p-subgroup. Let Δ be a collection of subgroups of S. Then, (G, Δ, S) is a locality.

Example

1) Let G be a finite group and S its Sylow p-subgroup. Let Δ be a collection of subgroups of S. Then, (G, Δ, S) is a locality.
2) Let $(\mathcal{L}, \Delta, S)$ be a locality and let $T \leq S$ be a 'well-chosen' subgroup.

Example

1) Let G be a finite group and S its Sylow p-subgroup. Let Δ be a collection of subgroups of S. Then, (G, Δ, S) is a locality.
2) Let $(\mathcal{L}, \Delta, S)$ be a locality and let $T \leq S$ be a 'well-chosen' subgroup. Set $\Gamma = \{N_P(T) | T \leq P \in \Delta\}$. Then, if $\Gamma \subset \Delta$, we have that $(N_{\mathcal{L}}(T), \Gamma, N_S(T))$ is a locality.

Example

1) Let G be a finite group and S its Sylow p-subgroup. Let Δ be a collection of subgroups of S. Then, (G, Δ, S) is a locality.
2) Let $(\mathcal{L}, \Delta, S)$ be a locality and let $T \leq S$ be a 'well-chosen' subgroup. Set $\Gamma = \{N_P(T) | T \leq P \in \Delta\}$. Then, if $\Gamma \subset \Delta$, we have that $(N_{\mathcal{L}}(T), \Gamma, N_S(T))$ is a locality.
3) Let $(\mathcal{L}, \Delta, S)$ be a locality and let $\mathcal{N}$ be a partial normal subgroup. Then, $\mathcal{N}$ is not necessarily a locality.

Work in progress

1. C. Broto and A. González gave a structure of simplicial sets to partial groups providing a topological point of view.

Work in progress

1. C. Broto and A. González gave a structure of simplicial sets to partial groups providing a topological point of view.
2. A. Chermak has defined a category of *regular* localities. In particular, partial normal subgroups and quotients of regular localities are again regular localities.

Work in progress

1. C. Broto and A. González gave a structure of simplicial sets to partial groups providing a topological point of view.
2. A. Chermak has defined a category of *regular* localities. In particular, partial normal subgroups and quotients of regular localities are again regular localities.
3. A. González has developed an extension theory for localities.

Work in progress

1. C. Broto and A. González gave a structure of simplicial sets to partial groups providing a topological point of view.
2. A. Chermak has defined a category of *regular* localities. In particular, partial normal subgroups and quotients of regular localities are again regular localities.
3. A. González has developed an extension theory for localities.

Questions:

1. What should a (partial) action of a partial group on a set be?

Work in progress

1. C. Broto and A. González gave a structure of simplicial sets to partial groups providing a topological point of view.
2. A. Chermak has defined a category of *regular* localities. In particular, partial normal subgroups and quotients of regular localities are again regular localities.
3. A. González has developed an extension theory for localities.

Questions:

1. What should a (partial) action of a partial group on a set be?
2. Is there any relation between two locality structures $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}, \Delta', S')$ on the same partial group $\mathcal{L}$?

Bibliography

There are some notes from Markus Linckelmann and Sejong Park available called 'Introduction to fusion systems'.
http://www.maths.nuigalway.ie/ park/papers/intro-fusion-systems.pdf
http://web.mat.bham.ac.uk/C.W.Parker/Fusion/fusion-intro.pdf

Bibliography

There are some notes from Markus Linckelmann and Sejong Park available called 'Introduction to fusion systems'.
http://www.maths.nuigalway.ie/ park/papers/intro-fusion-systems.pdf
http://web.mat.bham.ac.uk/C.W.Parker/Fusion/fusion-intro.pdf

- M. Aschbacher,R. Kessar, B. Oliver, *Fusion Systems in Algebra and Topology*, London Mathematical Society Lecture Note Series, 391. Cambridge Univ. Press, Cambridge, 2011.
- C. Broto, R. Levi, B. Oliver *The Homotopy Theory of Fusion Systems*, J. Amer. Math. Soc., 16, 779?856 (2003).
- A. Chermak *Fusion systems and Localities*, Acta Mathematica, Volume 211, Issue 1, pp 47-139.

THANK YOU VERY MUCH FOR YOUR ATTENTION!