Topologie algébrique
Série 12

May 15, 2017

Hand in exercise 3 on May 23.

1. (a) Define persistence module homomorphisms in a sensible way.
   (b) Verify that persistence modules form a category.
   (c) Define the direct sum of two persistence modules in a sensible way.
   (d) A persistence module $F : \mathbb{R} \to \mathrm{Vect}_{\mathbb{F}}$ is said to be *indecomposable* if the only way it can be written as a direct sum is $F \oplus 0$ (up to permutation). Show that interval modules $I_{[b,d)}$ (and any other interval variant) are indecomposable.

2. (a) Verify that the Vietoris–Rips complex from série 6 defines a filtration.
   (b) Give an equivalent definition of the Vietoris–Rips complex of a set of points in $\mathbb{R}^n$ as a flag complex.

If persistent homology is to be used to study data from real measurements, we should be a little bit worried. If the input measurements – for example the location of points in space – are subject to noise and have some "small error", it might be the case that we end up with "big error" in the persistence modules. This is the *stability question* that will be dealt with below.

Definition 1. Let F and G be tame persistence modules (i.e. persistence modules for which we have the interval decomposition). Write $PD(F)$ and $PD(G)$ for their persistence diagrams. The *bottleneck distance* between between $PD(F)$ and $PD(G)$ is

$$d_{\text{bottle}}(PD(F), PD(G)) = \inf_{\substack{\mu : PD(F) \to PD(G) \\ \text{bijection}}} \sup_{x \in PD(F)} \|x - \mu(x)\|_\infty .$$

Definition 2. Two persistence modules F and G are said to be *ε-interleaved* if

$$
\begin{array}{ccccc}
F(\alpha-\varepsilon) & \longrightarrow & & & F(\alpha'+\varepsilon) \\
& \searrow & & \nearrow & \\
& G(\alpha) & \longrightarrow & G(\alpha') &
\end{array}
\qquad
\begin{array}{ccccc}
& F(\alpha+\varepsilon) & \longrightarrow & F(\alpha'+\varepsilon) & \\
\nearrow & & \nearrow & & \\
G(\alpha) & \longrightarrow & G(\alpha') & &
\end{array}
$$

$$
\begin{array}{ccccc}
& F(\alpha) & \longrightarrow & F(\alpha') & \\
& \nearrow & & \searrow & \\
G(\alpha-\varepsilon) & \longrightarrow & & & G(\alpha'+\varepsilon)
\end{array}
\qquad
\begin{array}{ccccc}
F(\alpha) & \longrightarrow & F(\alpha') & & \\
\searrow & & \searrow & & \\
& G(\alpha+\varepsilon) & \longrightarrow & G(\alpha'+\varepsilon) &
\end{array}
$$

commute for all $\alpha \leq \alpha'$.

Theorem 3 (Stability)**.** *If F and G are tame and ε-interleaved persistence modules, then* $d_{bottle}(PD(F), PD(G)) \leq \varepsilon$.

3. (a) Verify that d_{bottle} is a metric (of the kind that may take value ∞) on tame persistence modules.
   (b) Let X be a topological space with $f, g : X \to \mathbb{R}$ nice enough functions that $PH_p(X^f)$ and $PH_p(X^g)$ are tame persistence modules.[1] Show that

   $$d_{\text{bottle}}\left(PD(PH_p(X^f)), PD(PH_p(X^g))\right) \leq \|f-g\|_\infty.$$

   This shows that a small "measuring error" in the function defining the filtration results in a small change in the persistence diagrams.

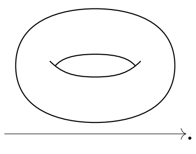
4. In the lectures I defined the Čech filtration in terms of open balls. We avoid some minor annoyances by considering closed balls instead, so consider the definition hereby changed!

   (a) Show that that for any ε there exists a finite set of points $P \subseteq \mathbb{R}^n$ such that

   $$d_{\text{bottle}}\left(PD(PH_p(VR(P))), PD(PH_p(\check{C}(P)))\right) \geq \varepsilon.$$

   (b) Let P be a finite subset of some metric space. Let $I_{[b,d)}$ is a summand in either $PH_k(VR(P))$ or $PH_k(\check{C}(P))$, with $d > 3b$, then it is a summand in the other persistence module also.[2]

5. Let X be a torus embedded in in $\mathbb{R}^3$. Let $f : X \to \mathbb{R}$ be projection onto the shown axis in

   Draw $PD(PH_k(X^f))$ for $k = 0, 1, 2$.

6. *(Not part of curriculum.)* Write a computer program that computes $H_*(K; \mathbb{Z}/2\mathbb{Z})$ for finite simplicial complexes K.

[1] For example, if you are familiar with such things, f a Morse function on a manifold M suffices. This is not relevant for the exercise.

[2] If $P \subseteq \mathbb{R}^n$, the factor 3 can be replaced with a smaller one.