A Generalization of the Boltzmann Distribution & Hodge theory

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Outline

- Consider an electrical circuit, represented by a circular wire ($M = S^1 \times D^2$) attached to a battery.
- The *current at* α is the number of charged particle crossings at an oriented cross-section α of the wire, per unit time.

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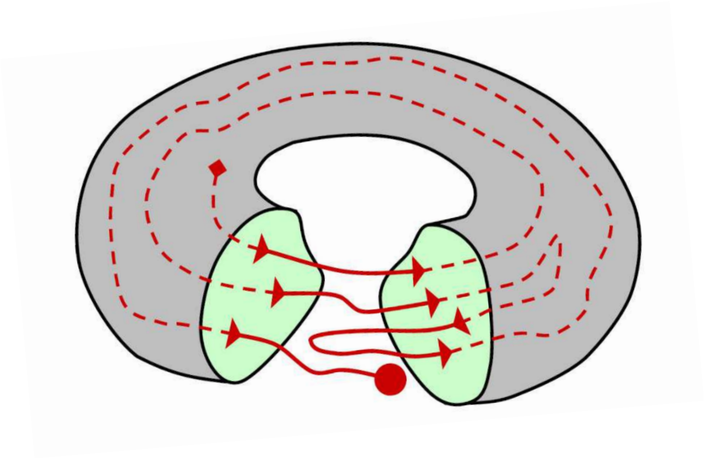
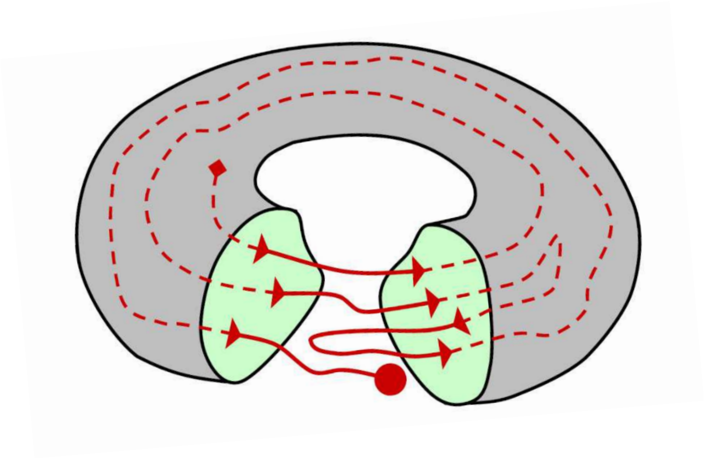
- For a single electron, the contribution to the current is $\omega_\alpha = \frac{1}{t} N$, where $N = N_+ - N_-$. If $\eta : S^1 \to M$ is the trajectory, then $\omega = [\eta] t^{-1} \in H_1(M; \mathbb{R})$.

An example





We're interested in studying these for general closed submanifolds $\eta : K \to M$ under some stochastic vector field.

Let X be a graph, with edges X_1 and vertices X_0. Suppose we're given functions

$$E : X_0 \to \mathbb{R} \qquad W : X_1 \to \mathbb{R}$$

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thought of as energies (or resistances) associated to each vertex and edge. Form the *master operator*

$$H = -\partial e^{-\beta \hat{W}} \partial^* e^{\beta \hat{E}} : C_0(X; \mathbb{R}) \to C_0(X; \mathbb{R})$$

where $\beta > 0$ is a noise (temperature) factor. Consider the *master equation*

$$\frac{dp}{dt} = Hp$$

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- If $M_X = \{E, W\}$, then $\gamma : [0, \tau] \to M_X$ does produce non-trivial currents.
- If $\gamma(0) = \gamma(\tau)$, then letting $\tau \to \infty$, we have

$$\text{average current} \;\; Q(\gamma) = \int_0^1 \sum_{k \in X_1} A^k(\gamma, \dot{\rho}^B) dt$$

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- A is the solution to the Kirchhoff problem, and ρ^B is the Boltzmann distribution.
- A can be expressed as a sum of spanning trees and ρ^B as a sum over the vertices of X.

- Consider a graph X with $W \equiv 0$.
- The steady state solution $\dot{p} = Hp = -\partial\partial^* e^{\beta\hat{E}} = 0$ yields $p_j = e^{-\beta E_j} j$. Normalizing, we obtain the classical Boltzmann distribution

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- Define a modified inner product on $C_0(X; \mathbb{R})$, $\langle x, y \rangle_E = e^{\beta E_x} \langle x, y \rangle$. We see that ρ^B is *closed*, and *co-closed* in the modified inner product;

$$\langle \rho^B, \partial y \rangle_E = \langle \partial^* \rho^B, y \rangle = 0,$$

so ρ^B is orthogonal to all boundaries in the modified inner product.

Theorem (Hodge)

There exists a decomposition of the k-forms on a Riemannian manifold

$$\Omega^k(M) \cong \Delta_k(M) \bigoplus B^k(M) \bigoplus B_k(M)$$
$$\cong \textit{harmonic} \bigoplus \textit{co-exact} \bigoplus \textit{exact}$$

where

- $B_k := \mathrm{im}(d : C^{k-1}(M) \to C^k(M))$
- $B^k := \mathrm{im}(d^* : C^{k+1}(M) \to C^k(M))$
- $\Delta_k(M) = \ker d \cap \ker d^*$

Let X be a connected CW complex of dimension d.
Fix $W : X_d \to \mathbb{R}$ and $E : X_{d-1} \to \mathbb{R}$.

Definition

A spanning tree for X is a subcomplex T such that

- $H_d(T; \mathbb{Z}) = 0$,
- $\beta_{d-1}(T) = \beta_{d-1}(X)$, *where $\beta_k(X)$ denotes the k-th Betti number,*
- $X^{(d-1)} \subset T$, *where $X^{(k)}$ is the k-skeletion of X.*

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Let θ_T denote the order of the torsion subgroup of $H_{d-1}(T;\mathbb{Z})$ and define the *weight of T* to be the positive real number

$$w_T := \theta_T^2 \prod_{b \in T_d} W(b)^{-1} .$$

Definition

For a spanning tree T of X, define a linear transformation

$$\bar{T} : C_d(X;\mathbb{R}) \to Z_d(X;\mathbb{R}).$$

For a d-cell b: if b is contained in T then $\bar{T}(b) = 0$. Otherwise, let c generate $Z_d(T \cup b) = H_d(T \cup b)$. Set $t_b = \langle c, b \rangle$. Then $\bar{T}(b) := c/t_b$, is a real d-cycle of X.

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Theorem (C, Chernyak, Klein)

The orthogonal projection $C_d(X;\mathbb{R})_W \to Z_d(X;\mathbb{R})$ is given by

$$A = \frac{1}{\Delta} \sum_T w_T \bar{T} \,,$$

where the sum is over all spanning trees, and $\Delta = \sum_T w_T$.

Definition

A spanning co-tree *for X is a subcomplex L such that*

- $i_* : H_{d-1}(L;\mathbb{R}) \cong H_{d-1}(X;\mathbb{R})$
- $\beta_{d-2}(L) = \beta_{d-2}(X)$.
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By definition we have

$$\begin{array}{ccc} Z_{d-1}(L) & \longrightarrow & H_{d-1}(L) \\ & \searrow^{\phi_L} & \downarrow \\ & & H_{d-1}(X) \end{array}$$

Define the *weight* of a spanning co-tree to be

$$\tau_L := |\operatorname{cok}\phi_L|^2 \prod_{b \in T_{d-1}} E(b)^{-1}$$

Spanning co-trees

For a spanning co-tree L of X, define ψ_L by the following diagram

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Theorem (C, Chernyak, Klein)

The splitting $H_{d-1}(X;\mathbb{R}) \to Z_{d-1}(X;\mathbb{R})$ is given by

$$\rho^B = \frac{1}{\tau} \sum_L \tau_L \psi_L$$

where the sum is over all spanning co-trees, and $\tau = \sum_L \tau_L$.

Combining these two splittings, we get

$$C_d(X;\mathbb{R})_W \cong Z_d(X;\mathbb{R}) \bigoplus B^d(X;\mathbb{R})$$

$$0 \longrightarrow Z_d(X;\mathbb{R}) \longrightarrow C_d(X;\mathbb{R}) \xrightarrow{\partial} B_{d-1}(X;\mathbb{R}) \longrightarrow 0 \qquad (A : C_d(X;\mathbb{R}) \dashrightarrow Z_d(X;\mathbb{R}))$$

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$$\begin{aligned} C_d(X;\mathbb{R})_W &\cong Z_d(X;\mathbb{R}) \bigoplus B^d(X;\mathbb{R}) \\ &\cong H_d(X;\mathbb{R}) \bigoplus B_d(X;\mathbb{R}) \bigoplus B^d(X;\mathbb{R}) \end{aligned}$$

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$$Q(\gamma) = \int_0^1 \sum_k A^k(\gamma, \dot{\rho}^B)dt$$

In the low temperature, adiabatic limit, we have the following:

Theorem (Chernyak, Klein, Sinitsyn)

For a connected graph X, the image of $Q : LM_X \to H_1(X; \mathbb{R})$ is contained the integral lattice $H_1(X; \mathbb{Z}) \subset H_1(X; \mathbb{R})$.

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Theorem (C, Chernyak, Klein)

Let X be a d-dimensional connected CW complex.

1. *Q can be written in the above form.*
2. *The image of $Q : LM_X \to H_d(X;\mathbb{R})$ is contained in $H_d(X;\mathbb{Z}[\frac{1}{D}])$, where D is determined by topological data.*