Splittings and products of topological abelian groups

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(Based on a joint work with M. J. Chasco, X. Dominguez and M. Tkachenko)

Throughout this talk I will prove that If G is a product of locally precompact abelian groups then every extension of topological abelian groups of the form $0 \to \mathbb{T}^\alpha \times \mathbb{R}^\beta \to X \to G \to 0$ splits.

This result is a form of the *splitting problem* and generalizes a result (proved by Moskowitz) that says that every extension of locally compact abelian groups the form $0 \to \mathbb{T} \to X \to G \to 0$ and $0 \to \mathbb{R} \to X \to G \to 0$ splits

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Preliminaries

- We will work on topological abelian groups *(abelian groups endowed with a topology making the maps $g \mapsto -g$ and $(g,h) \mapsto g+h$ continuous)*
- We will denote the unit circle by $\mathbb{T}$ and the real line by $\mathbb{R}$.
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- *An extension* of topological abelian groups:
  $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ short exact sequence [i, π relatively open continuous homomorphisms]

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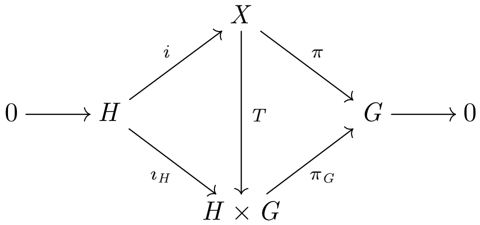

The Ext group

- We will call Ext(G, H) the set of equivalence classes of extensions of topological abelian groups of the form $0 \to H \to X \to G \to 0$. The set Ext$(G, H)$ with the operation induced by Baer sum in the equivalence classes of extensions of topological abelian groups is an abelian group. If every extension of topological abelian groups $0 \to H \to X \to G \to 0$ splits we will write Ext$(G, H) = 0$.
- The splitting problem consists in finding conditions on G and H so that Ext$(G, H) = 0$. It is known that if G is locally compact and H is $\mathbb{R}$ or $\mathbb{T}$ then Ext$(G, H) = 0$.

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Main Result

Theorem

Let $G = \prod_{i \in I} G_i$ *be a product of locally precompact abelian groups and let* α, β *be arbitrary cardinal numbers. Then*

$$\mathit{Ext}(G, \mathbb{T}^{\alpha} \times \mathbb{R}^{\beta}) = 0$$

We will say that a subgroup N of a topological abelian group G is admissible if there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open symmetric neighborhoods of the neutral element e in G such that $U_{n+1} + U_{n+1} + U_{n+1} \subset U_n$, for each $n \in \mathbb{N}$, and $N = \bigcap_{n\in\mathbb{N}} U_n$.

Lemma

Let M be $\mathbb{R}$ or $\mathbb{T}$ and let G be a topological abelian group. Suppose that $\mathcal{L}$ is a cofinal family of admissible subgroups of G ordered by inverse inclusion. Then

$$\mathit{Ext}(G/P, M) = 0 \ \forall P \in \mathcal{L} \implies \mathit{Ext}(G, M) = 0$$

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Sketch of the proof.

Step 1. Let M be $\mathbb{R}$ or $\mathbb{T}$. Show that it suffices to prove that $\mathrm{Ext}(G, M) = 0$ for every $G = \prod_{i \in I} G_i$ product of locally compact abelian groups using the properties:

- $\mathrm{Ext}(G, \prod_{i \in I} H_i) \cong \prod_{i \in I} \mathrm{Ext}(G, H_i)$
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Define $\mathcal{L}$ as the family of subgroups $N = \prod_{i \in I} N_i$ of $\prod_{i \in I} G_i$ such that

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Thank you for your attention!