Segal sections and categorical resolutions

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Segal objects

Segal Γ-spaces

Graeme Segal, *Categories and Cohomology Theories* (1974):
Denote by $\mathbf{Fin}_+$ the category of finite sets and partial maps.
For any finite set S, its elements $s \in S$ induce $\rho_s : S \to 1$ in $\mathbf{Fin}_+$. A Segal Γ-space is then a functor

$$\mathbf{Fin}_+ \xrightarrow{A} \mathbf{Top}$$

such that the *Segal maps*

$$A(S) \xrightarrow{\prod A(\rho_S)} A(1)^S$$

are (weak) homotopy equivalences for each $S \in \mathbf{Fin}_+$.

Homotopy coherent multiplication

For $S \in \mathbf{Fin}_+$, there is one more map $\pi_S : S \to 1$ defined on each element of S. Consider the span

$$\begin{array}{ccccc} & & A(S) & & \\ & \swarrow_{\prod A(\rho_S)}^{\sim} & & \searrow^{A(\pi_S)} & \\ A(1)^S & & & & A(1). \end{array}$$

Inverting the left map, we obtain multiplication operations $m_S : A(1)^S \to A(1)$. Indeed, in Ho **Top** the type $A(1)$ is a commutative monoid. But a Γ-space is more than an H-space (*Segal's delooping machinery*).

(Thanks to P. Taylor for his diagrams package.)

h-Algebra without operads?

One can replace $\mathbf{Fin}_+$ with Δ^{op} — associative monoids, Segal spaces etc. One can also think of a category 'like' **Fin** (Batanin, Barwick et al.) for working with homotopical algebraic structures such as $\mathbf{E}_n$-algebras, but without the use of topological operads.

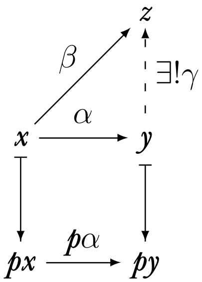
However, replace $(\mathbf{Top}, \times)$ with a symmetric monoidal category $(\mathcal{M}, \otimes)$, and Segal formalism stops working.

Sidestep (Segal, Lurie): $\mathcal{M}$ is a commutative monoid in **Cat**, hence a Γ-category.

Grothendieck (op)fibrations

opCartesian arrows (old school)

For $p : \mathcal{E} \to \mathcal{C}$, a morphism $\alpha : x \to y$ of $\mathcal{E}$ is p-opCartesian if any other $\beta : x \to z$ with $p\beta = p\alpha$ factors uniquely as $\beta = \gamma\alpha$, with $p(\gamma) = id_{p(y)}$:

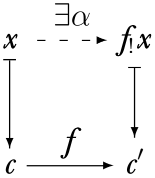


The definition is as in SGA1, today this is sometimes called locally opCartesian.

Opfibrations

A functor $p : \mathcal{E} \to \mathcal{C}$ is a Grothendieck opfibration if

1. For any $f : c \to c'$ in $\mathcal{C}$ and $x \in \mathcal{E}$ with $px = c$ there exists an opCartesian arrow $\alpha : x \to f_! x$ with $p\alpha = f$:



2. The composition of opCartesian arrows is opCartesian.

For $c \in \mathcal{C}$, denote $\mathcal{E}(c) = p^{-1}c$, the fibre over c. Then a choice of opCartesian arrows along $f : c \to c'$ defines a functor $f_! : \mathcal{E}(c) \to \mathcal{E}(c')$.

Symmetric monoidal cats as an example

Given a symmetric monoidal category $\mathcal{M}$ with $\otimes$, we construct an opfibration $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$.

- An object of $\mathcal{M}^{\otimes}$ is a pair of $S \in \mathbf{Fin}_+$ and an S-indexed family $\{X_s\}_{s\in S}$ of objects in $\mathcal{M}$.
- A map in $\mathcal{M}^{\otimes}$, $(S, \{X_s\}_{s\in S}) \to (T, \{Y_t\}_{t\in T})$, consists of $f : S \to T$ in $\mathbf{Fin}_+$ and a morphism $\otimes_{s\in f^{-1}(t)} X_s \to Y_t$ for each $t \in T$.
- The projection $(S, \{X_s\}_{s\in S}) \to S$ defines a functor $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$.

Note that $\mathcal{M}^{\otimes}(S) \cong \mathcal{M}^S$. The assignment $S \mapsto \mathcal{M}^{\otimes}(S)$ can be made into a pseudofunctor satisfying Segal conditions in $\mathbf{Cat}$.

Algebras as sections

For an opfibration $p : \mathcal{E} \to \mathcal{C}$, a section is a functor $A : \mathcal{C} \to \mathcal{E}$ with $pA = id$. Sections form a category $\mathsf{Sect}(\mathcal{C}, \mathcal{E})$ with fibrewise natural transformations.

In the case of $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$, consider a section $A : \mathbf{Fin}_+ \to \mathcal{M}^{\otimes}$ such that for each partial map $j : S \to T$ of the form

$$S \hookleftarrow T \xrightarrow{id} T,$$

the map $A(j)$ is opCartesian. Then $A(S) \cong (A(1), \ldots, A(1))$ and we get morphisms $A(1)^{\otimes S} \to A(1)$ in $\mathcal{M}^{\otimes}(1) = \mathcal{M}$. In this way, $A(1)$ becomes a commutative monoid in $\mathcal{M}$.

Segal conditions?

If $\mathcal{M}$ has weak equivalences $\mathcal{W}$ (assume preservation by $\otimes$), we still have no Segal-like description of monoid objects in $\mathcal{M}$. A usual section A of $\mathcal{E} \to \mathcal{C}$ sends $f : c \to c'$ to, in effect, a map $f_! A(c) \to A(c')$. Can one have a 'weak section' V, sending a map $f : c \to c'$ to a span

$$f_! V(c) \xleftarrow{\;\sim\;} V(f) \longrightarrow V(c')$$

with the left arrow in $\mathcal{W}$?

Simplicial replacements and Segal sections

Simplicial replacements

Aldridge Bousfield, Daniel Kan, *Homotopy limits, completions and localizations* (1972).

For a category $\mathcal{C}$, its simplicial replacement is a category $\mathbb{C}$ with

- An object $\mathbf{c}_{[n]} \in \mathbb{C}$ is a sequence of composable arrows

$$\mathbf{c}_{[n]} = c_0 \to c_1 \to \ldots \to c_n$$

- A map $\alpha : \mathbf{c}_{[n]} \to \mathbf{c}'_{[m]}$ consists of $a : [m] \to [n]$ in Δ such that $c_{a(i)} = c'_i$ for each $i \in [m]$.

As it is, $\mathbb{C} = (\int N\mathcal{C})^{\mathsf{op}}$. The assignments $\mathbf{c}_{[n]} \mapsto c_0$ or c_n define functors $\mathbb{C} \xrightarrow{h} \mathcal{C}$ and $\mathbb{C} \xrightarrow{t} \mathcal{C}^{\mathsf{op}}$.

Functors from the simplicial replacement

For a map $f : c \to c'$ in $\mathcal{C}$ can be viewed as on object of $\mathbb{C}$. Note the span in $\mathbb{C}$, $c \leftarrow (c \xrightarrow{f} c') \to c'$. For $F : \mathbb{C} \to \mathcal{N}$ we thus have diagrams like

$$\begin{array}{ccccc} & & F(c \to c') & & \\ & \swarrow & & \searrow & \\ F(c) & & & & F(c'). \end{array}$$

We can demand that the left arrow (and its likes coming from $\beta : \mathbf{c} \to \mathbf{d}$ with $c_{b(0)} = d_0$) is an isomorphism, and then one obtains a functor $\bar{F} : \mathcal{C} \to \mathcal{N}$. We can also ask the left arrow to be a weak equivalence if $\mathcal{N}$ has such.

Pulling opfibrations to $\mathbb{C}$

Remember the final object assignment $t : \mathbb{C} \to \mathcal{C}^{\mathrm{op}}$, $\mathbf{c}_{[n]} \mapsto c_n$. We want to use it to lift an opfibration (covariant family) $\mathcal{E} \to \mathcal{C}$ to $\mathbb{C}$. However, for this we need to replace it with the transpose (dual) *fibration* (contravariant family) $\mathcal{E}^{\top} \to \mathcal{C}^{\mathrm{op}}$.

It is characterised by the facts that $\mathcal{E}^{\top}(c) \cong \mathcal{E}(c)$ and that for each map $l : c' \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$, the transition functor $\mathcal{E}^{\top}(c) \to \mathcal{E}^{\top}(c')$ is isomorphic to $l_! : \mathcal{E}(c) \to \mathcal{E}(c')$. It is a fibration, however, so a section of $(\mathcal{M}^{\otimes})^{\top} \to \mathbf{Fin}_{+}^{\mathrm{op}}$ is a *coalgebra* in $\mathcal{M}$.

We then define $\mathbf{E} \to \mathbb{C}$ to be the pullback of $\mathcal{E}^{\top} \to \mathcal{C}^{\mathrm{op}}$ along $t : \mathbb{C} \to \mathcal{C}^{\mathrm{op}}$.

Segal sections

For an opfibration $\mathcal{E} \to \mathcal{C}$, a presection of it is a section $V : \mathbb{C} \to \mathbf{E}$ of the fibration constructed before. For $f : c \to c'$ ($f_! : \mathcal{E}(c) \to \mathcal{E}(c')$), we obtain a span in $\mathcal{E}(c')$ exactly as desired:

$$\begin{array}{ccc} & V(c \to c') & \\ \swarrow & & \searrow \\ f_! V(c) & & V(c'). \end{array}$$

If $\mathcal{E} \to \mathcal{C}$ has a homotopical structure (weak equivalences in $\mathcal{E}(c)$ preserved by $f_!$), then we can ask for the left arrow in the span to be a weak equivalence. V with such *Segal* conditions form a homotopical category $\mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E})$.

Categorical Resolutions

Categorical Resolutions of Singularities

A *categorical resolution* of a triangulated category $\mathcal{T}$ consists of an embedding (i.e., a *full and faithful* functor)

$$\mathcal{T} \xrightarrow{f} \mathcal{S}$$

into a 'good' triangulated category $\mathcal{S}$.

Theorem. Let Y be a separated scheme of finite type over k, $char(k) = 0$, then $\mathbf{D}(Y)$ admits a full and faithful functor

$$\mathbf{D}(Y) \xrightarrow{f} \mathcal{S}$$

so that $\mathcal{S}$ is smooth with a geometric semiorthogonal decomposition and f admits a 'good' right adjoint.

Topological Example

Take a finite CW-complex X of homotopy type $K(G, 1)$, e.g. G can be a braid group Br_n. Denote by NG the fundamental groupoid of X.

Take a regular cellular decomposition I of X. I is a partially ordered set ($\Rightarrow$ a category) by inclusion. The functor $F : I \to NG$ is obtained by sending each cell to its centre.

For any category $\mathcal{C}$, define $\mathbf{D}(\mathcal{C}, k)$ to be the derived category obtained from $Fun(\mathcal{C}, C^*(Vect_k))$. Then

$$F^* : \mathbf{D}(NG, k) \longrightarrow \mathbf{D}(I, k)$$

is full and faithful with characterisable essential image. We can thus study representations of G by passing to combinatorial objects over I.

Resolutions for Segal sections

Given a functor $F : \mathcal{D} \to \mathcal{C}$ and a homotopical opfibration $\mathcal{E} \to \mathcal{C}$, one gets a naturally induced weak equivalence preserving functor

$$F^* : \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \longrightarrow \mathbb{R}\mathsf{Sect}(\mathcal{D}, \mathcal{E}).$$

Questions one may ask: when is F^* homotopically fully faithful? What is its essential image?
For opfibrations like $\mathcal{M}^{\otimes} \to \mathbf{Fin}_+$, however, 'traditional' model-categorical techniques break down: transition functors $f_!$ in these opfibrations have no adjoints. In particular, no model structure on $\mathbb{R}\mathsf{Sect}$ in such cases.

Pushforward functor

Bousfield and Kan: simplicial replacements are to compute homotopy colimits through bar construction. Denote by $\mathsf{PSect}(\mathcal{C}, \mathcal{E}) = \mathsf{Sect}(\mathbb{C}, \mathbf{E})$ (no Segal condition).

Proposition. Given $\mathcal{E} \to \mathcal{C}$ as before (+ 'homotopy colimits in fibres') and $F : \mathcal{D} \to \mathcal{C}$, there are two homotopical functors

$$F^* : \mathsf{PSect}(\mathcal{C}, \mathcal{E}) \leftrightarrows \mathsf{PSect}(\mathcal{D}, \mathcal{E}) : F_!$$

and zigzags $F_! F^* \leftrightarrow id$, $id \leftrightarrow F^* F_!$. If $F_!$ preserves Segal sections, the zigzags become well-defined natural transformations on localisations, satisfying a triangle identity.

$$F_!(A)(\mathbf{c}_{[m]}) = |[n] \mapsto \coprod_{\mathbf{d}_{[n]}, \alpha : F(d_n) \to c_0} (f_m \dots f_1 \alpha)_! A(\mathbf{d}_{[n]})|$$

Resolutions of Segal section categories

Let $F : \mathcal{D} \to \mathcal{C}$ be an opfibration with the property that $N\mathcal{D}(c)$ is contractible for each $c \in \mathcal{C}$, $\mathcal{E} \to \mathcal{C}$ a homotopical opfibration with 'homotopy colimits in fibres' and weak equivalence preserving transition functors.

Theorem. The functor $F^* : \mathbb{R}\mathsf{Sect}(\mathcal{C}, \mathcal{E}) \to \mathbb{R}\mathsf{Sect}(\mathcal{D}, \mathcal{E})$ is full and faithful. Its essential image consists of Segal sections which, for each $c \in \mathcal{C}$, send all the maps in $\mathbb{D}(c)$ to weak equivalences.

Proof: (long) homotopy colimit manipulation, some similarities with Quillen Theorem A.

Outlook

- The theorem can be used to show in yet another way that there is an E_2-algebra structure on the Hochschild cochains of a dg-Algebra, with no mention of operads.
- One might also be interested in understanding more about the relation of the Segal section formalism to other model- and higher-categorical approaches.
- Specialising to algebra, one can define modules over Segal algebras and study their categories (triangulated structure?) and attempt to understand some deformation theory.

And the most important question...

Can one save Eurozone using Segal objects formalism?