# Topological Data Analysis on Data With Non-Symmetric Distances

YTM 2015

Scott Balchin

# Motivation

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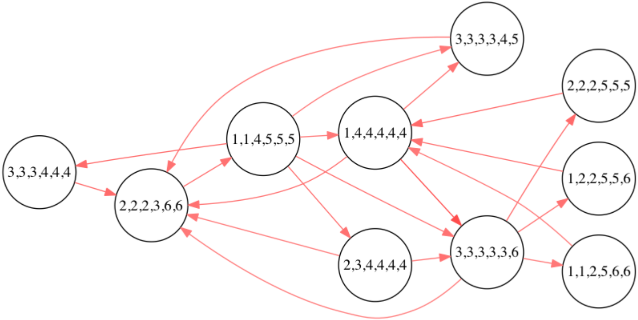
How can we study the structure of a space such as the following:



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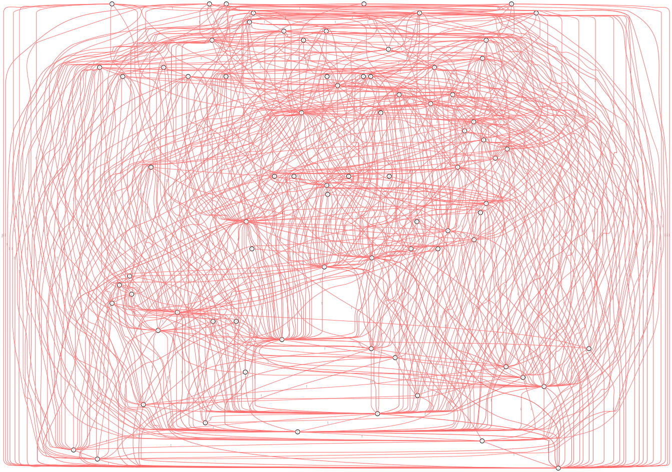
What about when we move a single dimension up?



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Is there a way to study these spaces and the relation between them using topological data analysis?

# General Philosophy of Topological Data Analysis

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- Point cloud of data in $\mathbb{R}^n$.
- Convert this point cloud into a family of topological spaces.
- Tools such as persistent homology.

# Vietoris-Rips Complex

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**Definition (Vietoris-Rips Complex)**

Given a finite collection of points $\{x_\alpha\}$ in $\mathbb{R}^n$ endowed with some metric, the *Vietoris-Rips complex* $\mathcal{R}_\epsilon$ is the abstract simplicial complex whose $k$-simplices correspond to unordered $(k+1)$-tuples of points $\{x_\alpha\}_0^k$ that are pairwise within distance $\epsilon$.

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The Vietoris-Rips complex is a *flag complex*, this means that its structure is determined solely by its edge structure.

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**Definition (Persistent Homology)**

Denote by $\mathcal{R}$ the full collection of Rips complexes constructed for a point cloud of data. For $i < j$, the $(i, j)$-*persistent homology* of $\mathcal{R}$ is the image of the induced homomorphism in homology $H_*(\mathcal{R}^i) \to H_*(\mathcal{R}^j)$.

In short, persistent homology allows us to track when homology generators are born and die as we vary our parameter $\epsilon$.

# Barcodes

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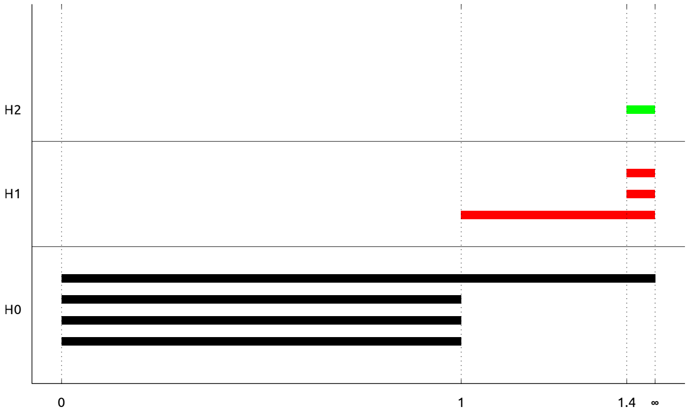
We can display the persistent homology data as a *persistence diagram* or *barcode* to visualise the results.



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- Not all data comes equipped with some canonical metric.
- Some data cannot be equipped with a metric without throwing away some sort of information.
- Main motivation : Directed graphs.
- How to construct a simplicial complex from a data set with a non-symmetric distance, which captures this non-symmetric features? (Joint work with Etienne Pillin)

# Non-Symmetric Complex Construction

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From now on we will assume that $\mathcal{X}$ is a data set with some distance $d$ between all points $X$ and $Y$ (possibly $\infty$). Without loss of generality assume

$$d_u(X, Y) = d(X, Y) \geqslant d(Y, X) = d_l(X, Y)$$

and let the *disparity* $\delta_{X,Y} = d_u(X, Y) - d_l(X, Y)$.

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Let $F(a, b) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing positive function in both variables, such that $F(a, 0) = 0$ for all $a$. For our purpose $a$ will be the dimension of a simplex and $b$ will take the values $\delta$.

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- We add a 3-simplex between points $X_1, \ldots, X_4$ if all $d_u(X_i, X_j) + F(1, \delta_{X_i, X_j}) \leqslant \epsilon$.

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- We add an $i$-simplex between points $X_1, \ldots, X_{i+1}$ if all $d_u(X_i, X_j) + F(i-2, \delta_{X_i,X_j}) \leqslant \epsilon$.

# Properties of the Non-Symmetric Complex

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If $d$ is symmetric, we will retrieve the classical Rips complex as we will have $\delta_{X,Y} = 0$ for all $X$ and $Y$, and because we asked for $F(a, 0) = 0$.

The complex is constructed so that if there is a large disparity $\delta_{X,Y}$, then the higher dimensional complexes involving the points $X$ and $Y$ will not be filled in until $\epsilon$ large. Can find near-symmetric nodes using this.

In the case that $\delta_{X,Y} = \infty$, then there will only ever be 1-simplicies whenever $X$ and $Y$ are involved.

We still get inclusions $\mathcal{N}_{\epsilon}^{F} \subset \mathcal{N}_{\delta}^{F}$ for $\epsilon < \delta$, which means we can do persistent homology.

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**Definition (Non-Symmetric Excess)**

Let $\delta_{\text{max}}$ be the maximum finite disparity between data points in $\mathcal{X}$. Then we define the *non-symmetric excess* $\mathcal{E}$ on a non-symmetric complex with respect to $F$ to be

$$\mathcal{E} = F(|\mathcal{X}| - 2, \delta_{\text{max}})$$

Where $|\mathcal{X}|$ is the number of data points.

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One way to overcome this would be truncating the construction, and allowing the higher dimensional simplicies be defined by the structure of the $i$ simplicies. This would be an *i-flag complex*.

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Other possible $F$ include:

- $F(a, b) = a^n b^m$ where $m, n \geqslant 1$
- $F(a, b) = b^a$
- $F(a, b) = a^b - 1$

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- It is a tool being adapted to sociology, anthropology, psychology, management, health, defence, etc.
- "It characterizes networked structures in terms of nodes (individual actors, people, or things within the network) and the ties or edges (relationships or interactions) that connect them."

# Directed Graphs via Twitter

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- Twitter has a naturally non-symmetric relation built in with its follower feature.
- We can represent this as a directed graph.
- We then can consider the shortest directed path between two people and let the distance between them be the length of this path.
- If no such path exists then we say the distance between the two people is $\infty$.

# Twitter's API

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- Twitter has a very convenient and practical API (Application Program Interface).
- This means we can actually create the graphs that we described with real data.
- We can start from an initial seed person and build their network (with some truncation).

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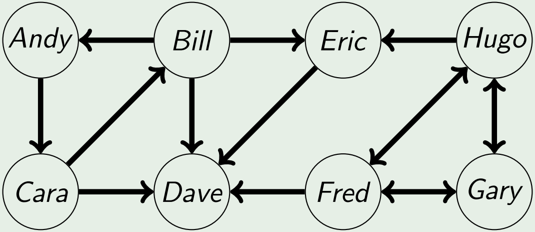

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The distance matrix with respect to the shortest path for this directed graph reads as below, where the entry $(i, j)$ is the distance from $i$ to $j$.

$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}
\begin{array}{c}
\begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{pmatrix}
0 & 2 & 1 & 2 & 3 & \infty & \infty & \infty \\
1 & 0 & 2 & 1 & 1 & \infty & \infty & \infty \\
2 & 1 & 0 & 1 & 2 & \infty & \infty & \infty \\
\infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\
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The only non-zero and non-infinite distance disparities are $\delta(A, B), \delta(B, C)$ and $\delta(A, C)$ which are all equal 1.

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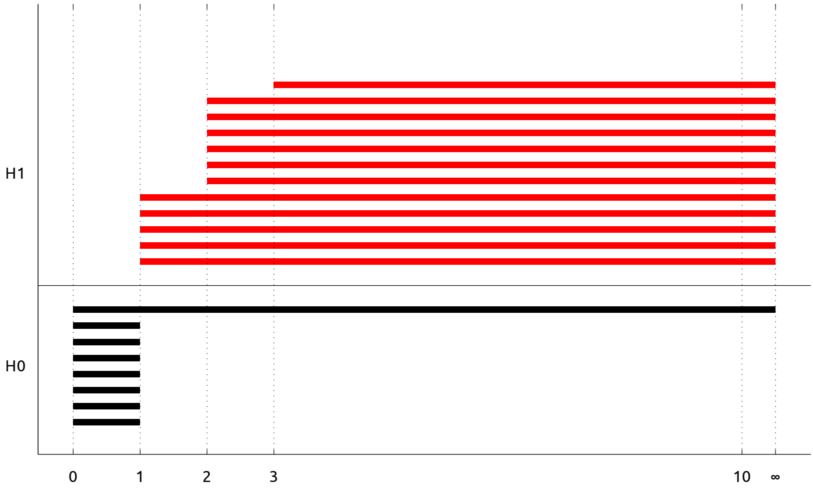

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Compare this with the matrix we would get by disregarding the directions.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 3 & 4 & 3 \\ 1 & 0 & 1 & 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 0 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 0 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 & 2 & 0 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 & 1 & 0 & 1 \\ 3 & 2 & 3 & 2 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

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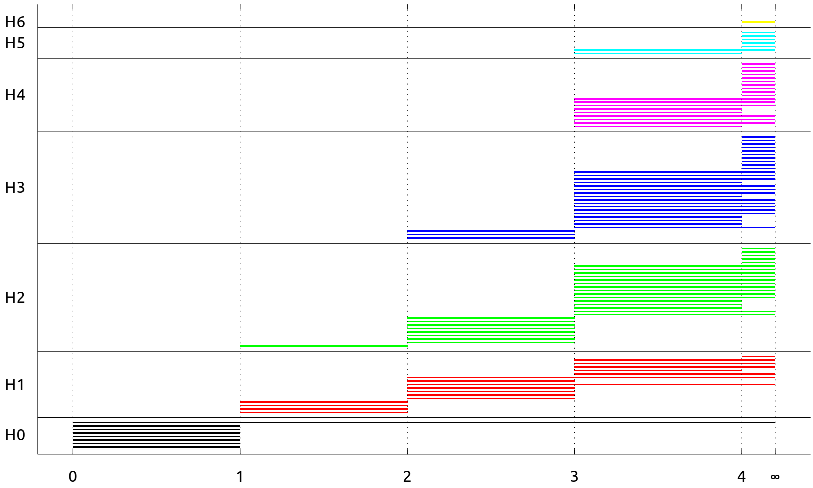

# A Brief Introduction

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- A *cycle of length $r$ of non-transitive dice* is an ordered collection of dice $(X_1, \ldots, X_r)$ such that:
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  1. $X_i \ggg X_{i+i}$ $\forall 1 \leqslant i \leqslant r - 1$
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- A dice $X$ is *triangular* if $d_1 + \cdots + d_n = T(n)$.

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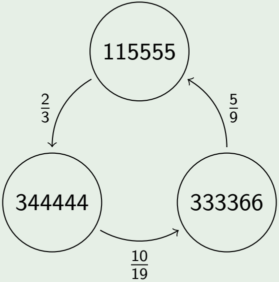

# Directed Graphs from Dice

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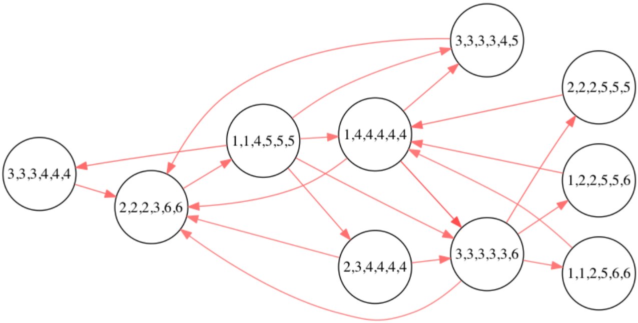
- Take some dice set $\mathcal{D}$.
- Consider all dice which appear in a non-transitive cycle.
- Plot with the dice being nodes, and the non-transitive relations being the directed edges.
- These graphs are always strongly connected.

# Triangular 6-Sided Dice



# Without Disregarding Distances

$$\begin{pmatrix} 0 & 3 & 3 & 3 & 4 & 2 & 4 & 3 & 1 & 4 \\ 3 & 0 & 3 & 3 & 4 & 2 & 4 & 3 & 1 & 4 \\ 3 & 3 & 0 & 3 & 1 & 2 & 1 & 2 & 1 & 1 \\ 3 & 3 & 1 & 0 & 2 & 2 & 2 & 3 & 1 & 2 \\ 4 & 2 & 2 & 4 & 0 & 3 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 0 & 2 & 1 & 2 & 2 \\ 4 & 2 & 2 & 4 & 3 & 3 & 0 & 1 & 2 & 3 \\ 3 & 1 & 1 & 3 & 2 & 2 & 2 & 0 & 1 & 2 \\ 2 & 2 & 2 & 2 & 3 & 1 & 3 & 2 & 0 & 3 \\ 4 & 2 & 2 & 4 & 3 & 3 & 3 & 1 & 2 & 0 \end{pmatrix}$$

# Without Disregarding Distances

- In this case we have no infinite distances.
- $\delta_{\text{max}} = 2$
- $\mathcal{E} = F(10 - 2, 2) = 8 \times 2 = 16$
- Therefore we should expect $\epsilon_{\text{max}} = 4 + 16 = 20$

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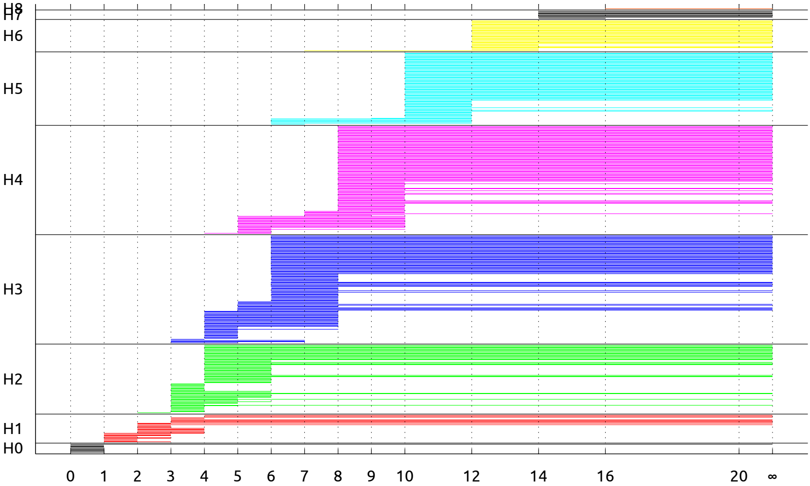

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