



Non-simple localizations of finite simple groups [☆]

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Abstract

Often a localization functor (in the category of groups) sends a finite simple group to another finite simple group. We study when such a localization also induces a localization between the automorphism groups and between the universal central extensions. As a consequence we exhibit many examples of localizations of finite simple groups which are not simple.

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Introduction

A group homomorphism $\varphi : H \rightarrow G$ is said to be a *localization* if and only if φ induces a bijection

$$\varphi^* : \text{Hom}(G, G) \cong \text{Hom}(H, G), \quad (0.1)$$

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where $\varphi^*(\psi) = \psi \circ \varphi$. This is an ad hoc definition which comes from [Cas, Lemma 2.1]. More details on localizations can be found there or in the introduction of [RST], where we exclusively study localizations of the form $H \hookrightarrow G$, where both H and G are simple groups. Due to the tight links with homotopical localizations, much effort has been dedicated to analyze which algebraic properties are preserved under localization. An exhaustive survey about this problem is nicely exposed in [Cas] by Casacuberta. For example, if H is abelian and $\varphi : H \rightarrow G$ is a localization, then G is again abelian. Similarly, nilpotent groups of class at most 3 are preserved (see [Asc] and the precursor [Lib2, Theorem 3.3] for class 2), but the question remains open for arbitrary nilpotent groups. Finiteness is not preserved, as shown by the example $A_n \rightarrow SO(n-1)$ (this is the main result in [Lib1]). In the present paper we focus on simplicity of finite groups and answer negatively a question posed both by Libman in [Lib2] and Casacuberta in [Cas] about preservation of simplicity. In these papers it was also asked whether perfectness is preserved. This is not the case either, as we show with totally different methods in [RSV].

Our main result here is that if $H \hookrightarrow G$ is a localization with H simple then G needs not be simple in general, see Corollary 1.7. There is, for example, a localization map from the Mathieu group M_{11} to the double cover of the Mathieu group M_{12} . This is achieved by a thorough analysis of the effect of a localization on the Schur multiplier, which encodes the information about the universal central extension. More precisely we prove the following theorem.

Theorem 1.5. *Let $i : H \hookrightarrow G$ be an inclusion of two non-abelian finite simple groups and $j : \tilde{H} \rightarrow \tilde{G}$ be the induced homomorphism on the universal central extensions. Assume that G does not contain any non-split central extension of H as a subgroup. Then $i : H \hookrightarrow G$ is a localization if and only if $j : \tilde{H} \rightarrow \tilde{G}$ is a localization.*

We only consider non-abelian finite simple groups since the localization of a cyclic group of prime order is either trivial or itself [Cas, Theorem 3.1]. Naturally, the second part of the paper deals with the effect of a localization on the outer automorphism group, which roughly speaking is dual to the Schur multiplier as it encodes the information about the “super-group” of all automorphisms. We first find a general criterion telling when an inclusion of automorphism groups is a localization (in the spirit of [RST, Theorem 1.4]). If we assume, moreover, that we start with a localization of finite simple groups, the conditions become quite elementary, see Theorem 2.4. However there exists even a more convenient set of conditions to check in practice.

Theorem 2.5. *Let $i : H \hookrightarrow G$ be a localization of non-abelian finite simple groups. The extension $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ is then a localization if the following two conditions are satisfied:*

- (1) $\text{Aut}(G) = \text{Aut}(H)G$.
- (2) $H = N_G(H)$.

These two conditions are possibly stronger than the set of necessary conditions mentioned earlier. We explain however in the final part of the paper that they are very close to be equivalent. In the particular case when both outer automorphism groups are cyclic of prime order they have the advantage to be easy to check.

Corollary 2.6. *Let $i : H \hookrightarrow G$ be a localization between two non-abelian finite simple groups. Assume that H is a maximal subgroup of G and that both $\text{Out}(H)$ and $\text{Out}(G)$ are cyclic groups of order p for some prime p . Then $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ is a localization.*

This yields many examples. Notice that the converse of the corollary does not hold: there exists a localization $\text{Aut}(L_2(7)) \hookrightarrow S_8$, but the induced morphism $L_2(7) \hookrightarrow A_8$ fails to be one, as we explain in Remark 2.7.

1. Preservation of simplicity

We first need to fix some notation. Let $\text{Mult}(G) = H_2(G; \mathbb{Z}) \cong H^2(G; \mathbb{C}^\times)$ denote the Schur multiplier of the finite simple group G and $\text{Mult}(G) \hookrightarrow \tilde{G} \twoheadrightarrow G$ be the universal central extension of G . In particular the only non-trivial endomorphisms of \tilde{G} are automorphisms. This is due to the fact that the only proper normal subgroups of \tilde{G} are contained in $\text{Mult}(G)$ and $\text{Hom}(G, \tilde{G}) = 0$ since the universal central extension is not split. For more details, a good reference is [Wei, Section 6.9]. Recall also that a group G is perfect if it is equal to its commutator subgroup. Equivalently G is perfect if $H_1(G; \mathbb{Z}) = 0$. If, moreover, $H_2(G; \mathbb{Z}) = 0$ we say that G is superperfect. Hence for a perfect group G we have that $\tilde{G} = G$ if and only if G is superperfect.

Is simplicity preserved under localization? We show that the answer is affirmative if H is maximal in G . By C_p we denote a cyclic group of order p .

Proposition 1.1. *Let G be a finite group and let H be a maximal subgroup which is simple. If the inclusion $H \hookrightarrow G$ is a localization, then G is simple.*

Proof. First notice that H cannot be normal in G . Indeed if H is normal, the maximality of H implies that the quotient G/H does not have any non-trivial proper subgroup. Hence $G/H \cong C_p$ for some prime p . But then G has a subgroup of order p and there is an endomorphism of G factoring through C_p , whose restriction to H is trivial. This contradicts the assumption that the inclusion $H \hookrightarrow G$ is a localization.

Thus, as H is simple, it contains no non-trivial normal subgroup of G . If G is not simple, a minimal normal non-trivial subgroup N of G is therefore a complement to H in G . But then both the identity map and the projection onto H with kernel N extend the inclusion $H \hookrightarrow G$. Therefore there are no normal proper non-trivial subgroups in G . \square

We indicate next (in Corollary 1.7) a generic situation where the localization of a simple group can be non-simple (it will actually be the universal cover of a simple group). To achieve this we study when a localization of finite simple groups induces a localization of the universal covers.

Proposition 1.2. *Let H and G be non-abelian finite simple groups. Assume that any homomorphism between the universal central extensions $\tilde{H} \rightarrow \tilde{G}$ sends $\text{Mult}(H)$ into $\text{Mult}(G)$. Then $p: \tilde{G} \twoheadrightarrow G$ and $q: \tilde{H} \twoheadrightarrow H$ induce an isomorphism $F: \text{Hom}(\tilde{H}, \tilde{G}) \xrightarrow{\cong} \text{Hom}(H, G)$ characterized by the property that $F(\tilde{\varphi})$ is the unique morphism $\varphi: H \rightarrow G$ such that $p \circ \tilde{\varphi} = \varphi \circ q$.*

Proof. First notice that p and q induce indeed a map $F: \text{Hom}(\tilde{H}, \tilde{G}) \rightarrow \text{Hom}(H, G)$. Let $\tilde{\varphi} \in \text{Hom}(\tilde{H}, \tilde{G})$. By hypothesis $\tilde{\varphi}(\text{Mult}(H)) \leq \text{Mult}(G)$, so $\text{Ker } q = \text{Mult}(H)$ is contained in $\text{Ker}(p \circ \tilde{\varphi})$. Hence, there exists a unique morphism $\varphi: H \rightarrow G$ such that $p \circ \tilde{\varphi} = \varphi \circ q$.

We show now that F is a bijection. Let $\alpha: H \rightarrow G$ be any homomorphism. Set $K = p^{-1}(\alpha(H))$ and $L = K^\infty$ the last term in the derived series of K . Then the restriction morphism $p: L \rightarrow \alpha(H)$ is a cover of $\alpha(H)$ since H is simple. There exists therefore a unique map from the universal cover $\tilde{\alpha}: \tilde{H} \rightarrow L$ such that $\alpha \circ q = p \circ \tilde{\alpha}$. Regarding $\tilde{\alpha}$ as a member of $\text{Hom}(\tilde{H}, \tilde{G})$, this is the unique element in $F^{-1}(\alpha)$. \square

The following corollary of Proposition 1.2 is a well-known consequence of the universal property of the universal central extension.

Corollary 1.3. *Let G be a non-abelian finite simple group and denote by $p: \tilde{G} \rightarrow G$ its universal central extension. Then we have an isomorphism $F: \text{Aut}(\tilde{G}) \xrightarrow{\cong} \text{Aut}(G)$.*

Of course an automorphism of the universal central extension does not always induce the identity on the center (all inner automorphisms do so). For example, let $G = L_3(7) = A_2(7)$, so $\tilde{L}_3(7) = SL_3(7)$ and $\text{Mult}(L_3(7)) = Z(SL_3(7)) \cong \mathbb{Z}/3$ is generated by the diagonal matrix D whose coefficients are 2's. There is an outer “graph automorphism” of order 2 given by the transpose of the inverse. It sends a matrix A to ${}^tA^{-1}$, so the image of D is D^{-1} .

Proposition 1.4. *Let G be a finite simple group. Then, the universal cover $\tilde{G} \rightarrow G$ is a localization.*

Proof. We have to show that $\tilde{G} \rightarrow G$ induces a bijection $\text{Hom}(G, G) \cong \text{Hom}(\tilde{G}, G)$ or equivalently, $\text{Aut}(G) \cong \text{Aut}(\tilde{G}) \setminus \{0\}$. This follows easily since the only non-trivial proper normal subgroups of \tilde{G} are contained in its center $\text{Mult}(\tilde{G})$. Thus any non-trivial homomorphism $\tilde{G} \rightarrow G$ can be decomposed as the canonical projection $\tilde{G} \rightarrow G$ followed by an automorphism of G . \square

Theorem 1.5. *Let $i: H \hookrightarrow G$ be an inclusion of two non-abelian finite simple groups and $j: \tilde{H} \rightarrow \tilde{G}$ be the induced homomorphism on the universal central extensions. Assume that G does not contain any non-split central extension of H as a subgroup. Then $i: H \hookrightarrow G$ is a localization if and only if $j: \tilde{H} \rightarrow \tilde{G}$ is a localization.*

Proof. Let $\varphi: \tilde{H} \rightarrow \tilde{G}$ be any group homomorphism. By composing with $p: \tilde{G} \rightarrow G$ we get a morphism $\tilde{H} \rightarrow G$. As G does not contain any subgroup isomorphic to a non-split central extension of H , $\varphi(\text{Mult}(\tilde{H})) \subset \text{Mult}(G)$. Hence Proposition 1.2 supplies bijections $F: \text{Hom}(\tilde{H}, \tilde{G}) \rightarrow \text{Hom}(H, G)$ and $F': \text{Hom}(\tilde{G}, \tilde{G}) \rightarrow \text{Hom}(G, G)$. Then the composition $F^{-1} \circ i^* \circ F'$ is a bijection if and only if i^* is so (if and only if i is a localization). Further, for any $\psi \in \text{Hom}(\tilde{G}, \tilde{G})$, Proposition 1.2 says that

$$F'(\psi) \circ i \circ q = F'(\psi) \circ p \circ j = p \circ \psi \circ j$$

and thus $F(j^*(\psi)) = F(\psi \circ j) = F'(\psi) \circ i = i^*(F'(\psi))$. Therefore the bijection $F^{-1} \circ i^* \circ F'$ is precisely j^* and we are done. \square

Remark 1.6. We do not know how to remove the assumption on the centers in Proposition 1.2. There exist morphisms between covers of finite simple groups which do not send the center into the center. One example is given in [CCN, p. 34] by the inclusion $\tilde{A}_5 \hookrightarrow U_3(5)$. A larger class of examples is obtained as follows: Let H be a finite simple group of order k and \tilde{H} its universal central extension of order $n = |\text{Mult}(H)| \cdot k$. The image of the representation $\tilde{H} \hookrightarrow S_n$ lies in A_n because the groups are perfect. Therefore A_n contains \tilde{H} . However we do not know of a single example of a localization $H \hookrightarrow G$ which does not satisfy this assumption and it is rather easy to check in practice.

Question. Let $i : H \hookrightarrow G$ be a localization. Is it possible that some subgroup of G is isomorphic to a non-trivial central extension of H ? If the answer is no, we would get a more general version of Theorem 1.5.

Beware that in general the induced morphism on the universal central extensions given by the above theorem is not an inclusion. For example, $L_2(11) \hookrightarrow U_5(2)$ is a localization by the main theorem in [RST]. However $U_5(2)$ is superperfect and the universal central extension $SL_2(11)$ of $L_2(11)$ is not a subgroup of $U_5(2)$. Nevertheless there is a localization $SL_2(11) \rightarrow U_5(2)$. The dual situation when H is superperfect leads to our counterexamples.

Corollary 1.7. *Let $i : H \hookrightarrow G$ be an inclusion of two non-abelian finite simple groups and assume that H is superperfect. Let also $j : H = \tilde{H} \hookrightarrow \tilde{G}$ denote the induced homomorphism on the universal central extensions. Then $i : H \hookrightarrow G$ is a localization if and only if $j : H \hookrightarrow \tilde{G}$ is a localization.*

Proof. There are no non-trivial central extensions of H so Theorem 1.5 applies. \square

Example 1.8. The inclusion $M_{11} \hookrightarrow \tilde{M}_{12}$ of the Mathieu group M_{11} into the double cover of the Mathieu group M_{12} is a localization. This follows from the above proposition. Note that M_{11} is not maximal in \tilde{M}_{12} (the maximal subgroup is $M_{11} \times C_2$), so this does not contradict Proposition 1.1. The following inclusions are localizations: $Co_2 \hookrightarrow Co_1$ and $Co_3 \hookrightarrow Co_1$ by [RST, Section 4]. As the smaller group is superperfect we get localizations $Co_2 \hookrightarrow \tilde{Co}_1$ and $Co_3 \hookrightarrow \tilde{Co}_1$.

We get many other examples of this type using [RST, Corollary 2.2]. All sporadic groups appearing in this corollary which have trivial Schur multiplier (that is $M_{11}, M_{23}, M_{24}, J_1, J_4, Co_2, Co_3, He, Fi_{23}, HN,$ and Ly) admit the double cover of an alternating group as localization (as $Mult(A_n)$ is cyclic of order 2 for $n > 7$).

Remark 1.9. The inclusion $Fi_{23} \hookrightarrow B$ of the Fischer group into the baby monster is a localization by [RST, Section 3(vi)]. This yields a localization $Fi_{23} \hookrightarrow \tilde{B}$. As the double cover \tilde{B} is a maximal subgroup of the Monster M , it would be nice to know if $\tilde{B} \hookrightarrow M$ is a localization. This would connect the Monster to the rigid component of the alternating groups (in [RST] we were able to connect all other sporadic groups to an alternating group by a zigzag of localizations, it is an open problem to determine whether or not any pair of finite non-abelian simple groups can be connected by a zigzag of localizations).

2. Localizations between automorphism groups

The purpose of this section is to show that a localization $H \hookrightarrow G$ can often be extended to a localization $Aut(H) \hookrightarrow Aut(G)$, similarly to the dual phenomenon observed in Theorem 1.5. This generalizes the observation made by Libman (cf. [Lib2, Example 3.4]) that the localization $A_n \hookrightarrow A_{n+1}$ extends to a localization $S_n \hookrightarrow S_{n+1}$ if $n \geq 7$. This result could be the starting point for determining the rigid component (as defined in [RST]) of the symmetric groups, but we will not go further in this direction. Let us recall two well-known results about the automorphism group of a finite simple group.

Lemma 2.1. *Let G be a non-abelian finite simple group. Then any proper normal subgroup of $\text{Aut}(G)$ contains G . In particular any endomorphism of $\text{Aut}(G)$ is either an isomorphism, or contains G in its kernel.*

Lemma 2.2. *Let G be a non-abelian finite simple group. Then any non-abelian simple subgroup of $\text{Aut}(G)$ is contained in G .*

Proof. Let H be a non-abelian simple subgroup of $\text{Aut}(G)$. The kernel K of the projection $\text{Aut}(G) \rightarrow \text{Out}(G)$ contains H because $\text{Out}(G)$ is solvable (this is the Schreier conjecture, whose proof depends on the classification of finite simple groups, see [GLS, Theorem 7.1.1]). \square

We consider from now on finite simple groups H and G , and their automorphism groups $\text{Aut}(H)$ and $\text{Aut}(G)$. Assume first that $\text{Aut}(H)$ is contained in $\text{Aut}(G)$ (any automorphism of H extends to one of G , see the discussion in [RST, Section 1]). We want to know when this is a localization (without claiming anything about $H \hookrightarrow G$ being a localization). The proof of the following theorem is very similar to that of [RST, Theorem 1.4].

Theorem 2.3. *Let $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ be an inclusion of the automorphism groups of two non-abelian finite simple groups H and G . Then j is a localization if and only if the following four conditions are satisfied:*

- (a) $\text{Aut}(G)$ acts transitively on the set Ω of subgroups of $\text{Aut}(G)$ isomorphic to $\text{Aut}(H)$.
- (b) $C_{\text{Aut}(G)}(\text{Aut}(H)) = 1$.
- (c) Any morphism $\psi : \text{Aut}(H)G/G \rightarrow \text{Aut}(G)$ extends uniquely to $\text{Out}(G) \rightarrow \text{Aut}(G)$.
- (d) If $\varphi : \text{Aut}(H) \rightarrow \text{Aut}(G)$ contains H in its kernel, then also $G \cap \text{Aut}(H) \leq \text{Ker } \varphi$.

Proof. Notice first that $j(H)$ must be contained in G by Lemma 2.2, so j restricts to an inclusion $H \hookrightarrow G$. Moreover, $\text{Aut}(G)$ is complete: $\text{Aut}(\text{Aut}(G)) = \text{Aut}(G)$ (see, for example, [Rot, Theorem 7.14]). An inclusion $\text{Aut}(H) \hookrightarrow \text{Aut}(G)$ (i.e. an element in Ω) extends to an automorphism in $\text{Aut}(G)$ if and only if condition (a) holds. Condition (b) holds if and only if this extension is unique. Let us now consider a non-injective morphism $\varphi : \text{Aut}(H) \rightarrow \text{Aut}(G)$, which must contain H in its kernel by Lemma 2.1. In this case it extends to $\text{Aut}(G)$ if and only if both condition (d) (the extension must contain G in its kernel) and condition (c) are satisfied. \square

Now we refine the above theorem, assuming that the inclusion $H \hookrightarrow G$ is a localization. As explained in [RST, Remark 1.3], the inclusion extends to an inclusion $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ given by the identification of $\text{Aut}(H)$ with $N_{\text{Aut}(G)}(H)$.

Theorem 2.4. *Let $i : H \hookrightarrow G$ be a localization of non-abelian finite simple groups. The extension $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ is then a localization if and only if conditions (c) and (d) above are satisfied.*

Proof. We only have to show that conditions (a) and (b) hold when i is a localization. As $\text{Aut}(H) = N_{\text{Aut}(G)}(H)$, the automorphism group $\text{Aut}(G)$ acts transitively on the set of subgroups of $\text{Aut}(G)$ isomorphic to H . Let A' be an element in Ω , i.e. a subgroup of $\text{Aut}(G)$ isomorphic to $\text{Aut}(H)$, and let K be its normal subgroup isomorphic to H . We know that there exists an

automorphism $\alpha \in \text{Aut}(G)$ such that $K^\alpha = H$, so conjugates A' to a subgroup $A = A'^\alpha$ isomorphic to $\text{Aut}(H)$ and containing H . Hence any element $x \in A$ normalizes H , so $A = \text{Aut}(H)$ because $\text{Aut}(H) = N_{\text{Aut}(G)}(H)$. The second condition is also satisfied since $C_{\text{Aut}(G)}(\text{Aut}(H))$ is a subgroup of the trivial group $C_{\text{Aut}(G)}(H)$. \square

As conditions (c) and (d) are not expressed in a suitable way to check in practice, we propose next a set of simpler conditions which guarantees $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ to be a localization. We are not quite sure that these conditions are also necessary and include below a discussion about this.

Theorem 2.5. *Let $i : H \hookrightarrow G$ be a localization of non-abelian finite simple groups. The extension $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ is then a localization if the following two conditions are satisfied:*

- (1) $\text{Aut}(G) = \text{Aut}(H)G$.
- (2) $H = N_G(H)$.

Proof. Condition (1) trivially implies condition (c) and condition (2) tells us that if x is an element in G such that conjugation by x is an automorphism of H , then x is actually an element of H . Thus condition (d) holds and we conclude by the preceding theorem. \square

The two conditions of the theorem imply in particular that the outer automorphism groups of H and G are isomorphic: $\text{Out}(H) \cong \text{Out}(G)$. We do not know of any localization between automorphism groups of finite simple groups which does not have this property. However even when $\text{Out}(H)$ and $\text{Out}(G)$ are cyclic of order 2, a localization $H \hookrightarrow G$ does not always induce one $\text{Aut}(H) \hookrightarrow \text{Aut}(G)$. For example, $i : L_3(3) \hookrightarrow G_2(3)$ is a localization (see [RST, Proposition 4.2]), but the normalizer in $\text{Aut}(L_3(3))$ of an $L_3(3)$ -subgroup of $G_2(3)$ is actually contained in $G_2(3)$. Thus $j : \text{Aut}(L_3(3)) \hookrightarrow \text{Aut}(G_2(3))$ cannot be a localization because that precomposing with j the non-trivial morphism $\text{Aut}(G_2(3)) \twoheadrightarrow \text{Out}(G_2(3)) \cong C_2 \hookrightarrow \text{Aut}(G_2(3))$ is trivial. The same phenomenon occurs again for $i : He \hookrightarrow Fi'_{24}$. Still, many examples can be directly derived from the following corollary of Theorem 2.5.

Corollary 2.6. *Let $i : H \hookrightarrow G$ be a localization between two non-abelian finite simple groups. Assume that H is a maximal subgroup of G and that both $\text{Out}(H)$ and $\text{Out}(G)$ are cyclic groups of order p for some prime p . Then $j : \text{Aut}(H) \hookrightarrow \text{Aut}(G)$ is a localization.*

Proof. As H is maximal in G , condition (2) is obviously satisfied. Moreover, $\text{Aut}(H)$ is not contained in G , so $G \text{Aut}(H)$ is a subgroup of $\text{Aut}(G)$ which strictly contains G . The index of G in $\text{Aut}(G)$ is prime, so condition (1) holds as well. \square

Directly from the corollary we deduce that $S_n \hookrightarrow S_{n+1}$ and $SL_2(p) \hookrightarrow S_{p+1}$ are localizations (by [RST, Proposition 2.3(i)] $L_2(p) \hookrightarrow A_{p+1}$ is a localization). Suzuki’s chain of groups $L_2(7) \hookrightarrow G_2(2)' \hookrightarrow J_2 \hookrightarrow G_2(4) \hookrightarrow \text{Suz}$ (see [Gor, pp. 108–109]) also extends to a chain of localizations of automorphism groups

$$\text{Aut}(L_2(7)) \hookrightarrow \text{Aut}(G_2(2)') \hookrightarrow \text{Aut}(J_2) \hookrightarrow \text{Aut}(G_2(4)) \hookrightarrow \text{Aut}(\text{Suz}).$$

Remark 2.7. The converse of the corollary is false in general. There exist localizations between automorphism groups which do not restrict to a localization of the corresponding finite simple groups. Consider, for example, the transitive action of $K = PGL_2(7) \cong \text{Aut}(L_2(7))$ on the projective line. Thus K is a subgroup of the symmetric group $S_8 = \text{Aut}(A_8)$. We check now that this is a localization with the help of Theorem 2.3 (and the information from the atlas [CCN]). As this is the unique equivalence class of representations of K of degree 8, S_8 acts transitively on its K -subgroups. Moreover, K contains an odd permutation, so $S_8 = KA_8$, which implies condition (c). Finally, condition (b) holds as well since the fixed set of the stabilizer of a point x of the projective line under the action of K is reduced to x (one could also check Eq. (0.1) quickly with the help of MAGMA). However the induced morphism $L_2(7) \hookrightarrow A_8$ fails to be a localization: there are three conjugacy classes of $L_2(7)$ -subgroups in A_8 , and only two of them fuse in S_8 .

We end the paper with a discussion on the two sets of conditions appearing in Theorems 2.3 and 2.5. We prove first that condition (d) is actually equivalent to condition (2). Consider the intersection

$$I_H = \bigcap \{ \text{Ker } \alpha \mid \alpha : \text{Out}(H) \rightarrow \text{Aut}(H) \}.$$

Proposition 2.8. *Let $i : H \hookrightarrow G$ be a localization of non-abelian finite simple groups and assume that $I_H = 1$. The following two conditions are then equivalent:*

- (d) *If $\varphi : \text{Aut}(H) \rightarrow \text{Aut}(G)$ contains H in its kernel, then also $G \cap \text{Aut}(H) \leq \text{Ker } \varphi$.*
- (2) $H = N_G(H)$.

Proof. We have seen in Theorem 2.5 that (2) always implies (d). Now if $I_H = 1$ the intersection of all morphisms $\varphi : \text{Aut}(H) \rightarrow \text{Aut}(G)$ containing H in their kernel is precisely H while condition (d) implies that $G \cap \text{Aut}(H)$ is contained in that intersection, hence $G \cap \text{Aut}(H) = H$. On the other hand, $N_G(H)$ consists in those automorphisms of H which are inner automorphisms of G , and we get (2). \square

Notice that I_H is always contained in the commutator subgroup $[\text{Out}(H), \text{Out}(H)]$: consider any element $x \in \text{Out}(H)$ which is not zero in the abelianization $\text{Out}(H)_{ab}$. We construct a morphism $\alpha_x : \text{Out}(H) \rightarrow \text{Aut}(H)$ such that $\alpha_x(x) \neq 1$ as a composition of type:

$$\text{Out}(H) \twoheadrightarrow \text{Out}(H)_{ab} \cong \bigoplus \mathbb{Z}/q \twoheadrightarrow \mathbb{Z}/q \hookrightarrow \text{Aut}(H),$$

where the cyclic component \mathbb{Z}/q is chosen so that the image of x is not zero. The inclusion $\mathbb{Z}/q \hookrightarrow \text{Aut}(H)$ is any such inclusion. This shows that $x \notin I_H$. In particular, when $\text{Out}(H)$ is abelian, I_H must be trivial. This takes care of all sporadic and alternating groups (the outer automorphism group is always trivial or cyclic of order 2, except for A_6 where it is a Klein group, see, for example, [GLS, Chapter 5]). Thus the only case where it could be that I_H is not trivial is that of the (twisted) Chevalley groups.

Proposition 2.9. *Let H be any non-abelian finite simple group. Then $I_H = 1$.*

Proof. We can suppose that H is a (twisted) finite Chevalley group from now on. We learn from [GLS, Theorem 2.5.12] that $\text{Aut}(H)$ is a split extension of $\text{Inndiag}(H)$ by a certain group $\Phi_H \Gamma_H$, and likewise $\text{Out}(H)$ is a split extension of $O = \text{Outdiag}(H)$ by the same group $\Phi_H \Gamma_H$. Therefore I_H must be contained in O . The only cases where O is not trivial are indicated in the quoted theorem: $A_m^\epsilon(q)$, $B_m(q)$, $C_m(q)$, $D_{2m}(q)$, ${}^2D_{2m}(q)$, $D_{2m+1}^\epsilon(q)$, $E_6^\epsilon(q)$, and $E_7(q)$.

When $H = A_m^\epsilon(q)$, the group O is cyclic of order $r = (m + 1, q - \epsilon)$ and $\text{Out}(H)$ is a semi-direct product of O with $\Phi_H \cong C_{p^s}$, a cyclic group of field automorphisms (or with $C_{p^s} \times C_2$ where C_2 is the graph automorphism group Γ_H in the case $\epsilon = +$). The group of field automorphisms acts faithfully on a cyclic group of order r inside a torus of H , so one can actually embed $\text{Out}(H)/\Gamma_H$ in $\text{Aut}(H)$. The case of the graph automorphism of order 2 is similar and therefore $I_H = 1$.

When H is one of the groups $B_m(q)$, $C_m(q)$, ${}^2D_{2m}(q)$, or $E_7(q)$ the group O is cyclic of order 2 and the outer automorphism group splits as a direct product $C_2 \times \Phi_H \Gamma_H$. So $O \cap [\text{Out}(H), \text{Out}(H)] = 1$ in these cases and I_H must be trivial.

When $H = D_{2m}(q)$, the group O is a Klein group, centralized by the field automorphisms Φ_H , but with a faithful action of the graph automorphisms Γ_H , isomorphic to Σ_3 or C_2 . As $H = P\Omega_{4m}(q, f)$ where f is the bilinear form $\sum_{i=1}^{2m} x_i x_{-i}$ (see [GLS, p. 71]), it contains a subgroup isomorphic to Σ_{2m} . Hence, one can construct a morphism $\text{Out}(H) \rightarrow \text{Aut}(H)$ which restricts to an injection on O and so $I_H = 1$.

When $H = D_{2m+1}^\epsilon(q)$, the group O is either cyclic of order 2 (in which case we conclude as in the second case), or of order 4. There exists then a quotient of $\text{Out}(H)$ of the form $C_4 : C_2 \cong D_8$ since the subgroup of $\Phi_H \Gamma_H$ acting trivially on C_4 has index 2. When $\epsilon = +$, we conclude as above. When $\epsilon = -$, $H = P\Omega_{4m+2}(q, f)$ where f is the bilinear form

$$\sum_{i=1}^{2m} x_i x_{-i} + x_{2m+1}^2 + b x_{2m+1} x_{-2m+1} + x_{-2m+1}^2,$$

so H contains a subgroup isomorphic to Σ_{2m} . In any case D_8 can be embedded into H .

When $H = E_6^\epsilon(q)$, the group O is cyclic of order 3 (or trivial). There exists then a quotient of $\text{Out}(H)$ of the form $C_3 : C_2 \cong \Sigma_3$, which always embeds into $D_5^\epsilon(q)$, a subgroup of H by [GLS, Table 4.5.2]. \square

Likewise, conditions (c) and (1) are very close to be equivalent.

Proposition 2.10. *Let $i : H \hookrightarrow G$ be a localization of non-abelian finite simple groups and assume that $\text{Out}(G)$ is nilpotent. The following two conditions are then equivalent:*

- (c) Any morphism $\psi : \text{Aut}(H)G/G \rightarrow \text{Aut}(G)$ extends uniquely to $\text{Out}(G) \rightarrow \text{Aut}(G)$.
- (1) $\text{Aut}(G) = \text{Aut}(H)G$.

Proof. We only have to prove that (c) implies (1). Let N be the normal closure of the subgroup $\text{Aut}(H)G/G \leq \text{Out}(G)$. If N were a strict subgroup of $\text{Out}(G)$, there would exist an abelian quotient of $\text{Out}(G)/N$ (recall that $\text{Out}(G)$ is solvable) which would yield a non-trivial morphism $\text{Out}(G) \rightarrow \text{Aut}(G)$ extending the trivial one from $\text{Aut}(H)G/G$, contradicting condition (c). Thus $N = \text{Out}(G)$. But if $\text{Out}(G)$ is nilpotent, any maximal subgroup is normal, so $\text{Aut}(H)G/G = \text{Out}(G)$ and condition (1) holds. \square

Question. Is it true that conditions (1) is equivalent to (c)? If it were so, Theorem 2.5 would characterize the localizations $H \hookrightarrow G$ which extend to a localization $\text{Aut}(H) \hookrightarrow \text{Aut}(G)$ by a very manageable criterion.

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