Waldhausen K-theory via comodules

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Outline

Overview

Background

Let K denote the functor from Waldhausen categories to spectra.

Let $\mathrm{R}_X^{\mathrm{hf}}$ denote the category of homotopically finite retractive spaces over a simplicial set X.

- $A(X)$ is the K-theory spectrum of the Waldhausen category $\mathrm{R}_X^{\mathrm{hf}}$, with cofibrations and weak equivalences created in the underlying category of simplicial sets.
- It is well known that
$$A(X) \simeq K\left(\mathrm{Mod}^{\mathrm{hf}}_{\Sigma^\infty(\Omega X)_+}\right).$$
- [Blumberg-Mandell, 2010] If X is simply connected, then
$$A(X) \simeq K\left(\mathrm{Mod}^{\mathrm{th}}_{DX}(\mathbf{S})\right).$$

THE MAIN THEOREM

A sort of Koszul dual of the module description of $K(X)$ on the previous slide...

THEOREM (H.-SHIPLEY, 2014)

For any connected simplicial set X, the categories

$$\mathrm{Comod}^{\mathrm{hf}}_{X_+} \quad \textit{and} \quad \mathrm{Comod}^{\mathrm{hf}}_{\Sigma^\infty X_+}$$

of homotopically finite X_+-comodules and $\Sigma^\infty X_+$-comodules admit Waldhausen category structures such that there are natural weak equivalences of K-theory spectra

$$A(X) \simeq K\left(\mathrm{Comod}^{\mathrm{hf}}_{X_+}\right) \simeq K\left(\mathrm{Comod}^{\mathrm{hf}}_{\Sigma^\infty X_+}\right).$$

Proof strategy

1. Establish existence of Quillen equivalence

$$(\mathrm{R}_X)_{\mathcal{E}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Comod}_{X_+})_{\mathcal{E}} \ ,$$

giving rise, via a result of Dugger and Shipley, to

$$K\big((\mathrm{R}_X)^{\mathrm{hf}}_{\mathcal{E}}\big) \simeq K\big((\mathrm{Comod}_{X_+})^{\mathrm{hf}}_{\mathcal{E}}\big).$$

2. Apply Hovey's stabilization machine to $(\mathrm{Comod}_{X_+})_{\mathcal{E}}$, obtaining a model category structure $(\mathrm{Comod}_{\Sigma^\infty X_+})_{\mathcal{E}}$ such that

$$K\big((\mathrm{Comod}_{X_+})^{\mathrm{hf}}_{\mathcal{E}}\big) \simeq K\big((\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{hf}}_{\mathcal{E}}\big).$$

Potential applications

Work in progres...

- A new description of the splitting of

$$A(X \times S^1) \simeq A(X) \times BA(X) \times (\text{nil terms}).$$

- A new description of the assembly map

$$A(*) \wedge X_+ \to A(X).$$

Conventions

- X is an unpointed simplicial set.
- $\mathcal{E}_*$ is a generalized, reduced homology theory.
- $(\mathrm{sSet}_*)_{\mathcal{E}}$ is the category of pointed simplicial sets endowed with the model category structure such that the weak equivalences are the $\mathcal{E}_*$-homology isomorphisms, while the cofibrations are the levelwise injections. The classes of weak equivalences, fibrations and cofibrations in $(\mathrm{sSet}_*)_{\mathcal{E}}$ are denoted
$$\mathcal{W}_{\mathcal{E}}, \mathcal{F}_{\mathcal{E}}, \text{ and } \mathcal{C}_{\mathcal{E}},$$
respectively.

Comodules and retractive spaces

X_+-COMODULES

Comod_{X_+} = category of right X_+-comodules in $(\mathsf{sSet}_*, \wedge, S^0)$

- Objects: pairs (Y, ρ), where $\rho : Y \to Y \wedge X_+$ is coassociative and counital.
- Comod_{X_+} is complete and cocomplete, as well as tensored, cotensored, and enriched over sSet_*.
- There is an sSet_*-adjunction

$$\begin{tikzcd} \mathsf{Comod}_{X_+} \arrow[r, shift left=2, "U"] \arrow[r, phantom, "\perp"] & \mathsf{sSet}_* \arrow[l, shift left=2, "F_{X_+}"] \end{tikzcd},$$

where $F_{X_+}(Y) = \big(Y \wedge X_+, Y \wedge (\Delta_X)_+\big)$ for all Y, and U is the forgetful functor.

X_+-COMODULES

- For any simplicial map $a : X' \to X$, there is a pushforward functor

$$a_* : \mathrm{Comod}_{X'_+} \to \mathrm{Comod}_{X_+},$$

specified on objects by

$$a_*(Y, \rho) = \big(Y, (Y \wedge a_+)\rho\big),$$

and which admits a right adjoint

$$a^* : \mathrm{Comod}_{X_+} \to \mathrm{Comod}_{X'_+}$$

that commutes with colimits.

- If (X, μ, x_0) is a simplicial monoid, then the monoidal structure $(\mathrm{sSet}_*, \wedge, S^0)$ lifts to a monoidal structure

$$\big(\mathrm{Comod}_{X_+}, \otimes, (S^0, \rho_u)\big),$$

which is symmetric if μ is commutative.

RETRACTIVE SPACES OVER X

$\mathrm{R}_X =$ category of retractive objects over X

- Objects: $X \xrightarrow{i} Z \xrightarrow{r} X$ such that $ri = \mathrm{Id}_X$.
- There is an adjunction

$$\begin{tikzcd} \mathrm{R}_X \arrow[r, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathrm{sSet}_* \arrow[l, shift left, "\mathrm{Ret}_X"] \end{tikzcd},$$

where

$$\mathrm{Ret}_X(Y, y_0) = X \xrightarrow[x \mapsto (y_0, x)]{i_{y_0}} Y \times X \xrightarrow{\mathrm{proj}_2} X$$

and

$$V(X \xrightarrow{i} Z \xrightarrow{r} X) = \big(Z/i(X), i(X)\big).$$

RETRACTIVE SPACES OVER X

- For any generalized reduced homology theory $\mathcal{E}_*$ and any $X \xrightarrow{i} Z \xrightarrow{r} X$, and any choice of basepoint in X,

$$\mathcal{E}_*(Z) \cong \mathcal{E}_*\big(Z/i(X)\big) \oplus \mathcal{E}_*(X).$$

- For any simplicial map $a : X' \to X$, there is an adjunction

$$\begin{tikzcd} \mathrm{R}_{X'} \arrow[r, shift left=2, "a_*"] \arrow[r, phantom, "\perp"] & \mathrm{R}_X, \arrow[l, shift left=2, "a^*"] \end{tikzcd},$$

given by pushout and pullback along a.

The key adjunction

Theorem (H.-Shipley, 2014)

There is an adjoint pair of functors, natural in X,

$$\begin{tikzcd} \mathrm{R}_X \arrow[r, shift left, "-/X"] \arrow[r, phantom, "\perp"] & \mathrm{Comod}_{X_+} \arrow[l, shift left, "-\star X"] \end{tikzcd},$$

preserving $\mathcal{E}_*$-equivalences and such that the counit map is a natural isomorphism and the unit map a natural $\mathcal{E}_*$-equivalence, for every generalized reduced homology theory $\mathcal{E}_*$.

Remark

When $X = *$, this is an equivalence of categories: R_* and Comod_{S^0} are equivalent to sSet_*, and $-/*$ and $-\star *$ induce the identity functors. It is not an equivalence if $X \neq *$.

The $-\star X$ functor

$$-\star X : \mathrm{Comod}_{X_+} \to \mathrm{R}_X : (Y,\rho) \mapsto (Y\star X, i_\rho, r_\rho)$$

where

$$\begin{tikzcd} Y\star X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y\times X \arrow[d, "\pi_Y"] \\ Y \arrow[r, "\rho"] & Y\wedge X_+ \end{tikzcd}$$

is a pullback in sSet, and

$$i_\rho : X \to Y\star X : x \mapsto (y_0, x) \quad \text{and} \quad r_\rho : Y\star X \to X : (y,x) \mapsto x.$$

Example

$F_{X_+}(Y)\star X = \mathrm{Ret}_X(Y).$

THE $-/X$ FUNCTOR

$$-/X : \mathsf{R}_X \to \mathsf{Comod}_{X_+} : (Z, i, r) \mapsto \big(Z/i(X), \rho_{(i,r)}\big),$$

where

$$\rho_{(i,r)} : Z/i(X) \to \big(Z/i(X)\big) \wedge X_+$$

is the unique pointed simplicial map such that

$$\begin{tikzcd} Z \arrow[r, "(p_i \times r)\Delta_Z"] \arrow[dr, "p_i"'] & \big(Z/i(X)\big) \times X \arrow[r] & \big(Z/i(X)\big) \wedge X_+ \\ & Z/i(X) \arrow[ur, dotted, "\rho_{(i,r)}"'] & \end{tikzcd}$$

commutes, where $p_i : Z \to Z/i(X)$ is the quotient map.

Model category structures

The unstable case

Theorem (H.-Shipley, 2014)

There are cofibrantly generated, left proper, simplicial model category structures $(\mathsf{R}_X)_{\mathcal{E}}$ *and* $(\mathsf{Comod}_{X_+})_{\mathcal{E}}$ *such that*

$$(\mathsf{R}_X)_{\mathcal{E}} \underset{-\star X}{\overset{-/X}{\rightleftarrows}} (\mathsf{Comod}_{X_+})_{\mathcal{E}}$$

is a Quillen equivalence and

- $\mathsf{WE}_{\mathsf{Comod}_{X_+}} = U^{-1}(\mathsf{WE}_{\mathcal{E}})$, $\mathsf{Cof}_{\mathsf{Comod}_{X_+}} = U^{-1}(\mathsf{Cof}_{\mathcal{E}})$, *and*
- $\mathsf{WE}_{\mathsf{R}_X} = V^{-1}(\mathsf{WE}_{\mathcal{E}})$, $\mathsf{Cof}_{\mathsf{R}_X} = V^{-1}(\mathsf{Cof}_{\mathcal{E}})$.

The existence of $(\mathsf{R}_X)_{\mathcal{E}}$ is standard (see below); we prove the existence of $(\mathsf{Comod}_{X_+})_{\mathcal{E}}$ in two complementary ways, by right- and left-induction.

REMARKS

- By a standard "slice" argument, R_X inherits a left proper, cofibrantly generated, simplicial model category structure from $(\mathrm{sSet})_{\mathcal{E}}$ such that $f : (Z, i, r) \to (Z', i', r')$ is a fibration (respectively, cofibration or weak equivalence) if and only if the underlying morphism of simplicial sets $f : Z \to Z'$ is of the same type.
- If $a : X' \to X$ is a simplicial map, then

$$\begin{tikzcd} (\mathrm{R}_{X'})_{\mathcal{E}} \arrow[r, shift left=2, "a_*"] \arrow[r, phantom, "\perp"] & (\mathrm{R}_X)_{\mathcal{E}} \arrow[l, shift left=2, "a^*"] \end{tikzcd},$$

  and

$$\begin{tikzcd} (\mathrm{Comod}_{X'})_{\mathcal{E}} \arrow[r, shift left=2, "a_*"] \arrow[r, phantom, "\perp"] & (\mathrm{Comod}_{X_+})_{\mathcal{E}} \arrow[l, shift left=2, "a^*"] \end{tikzcd},$$

  are Quillen pairs that are Quillen equivalences if a is an $\mathcal{E}_*$-equivalence.

PROOF BY RIGHT-INDUCTION

Apply standard transfer of cofibrantly generated model category structure to

$$\begin{tikzcd} \mathrm{R}_X \arrow[r, shift left=2, "-/X"] \arrow[r, phantom, "\perp"] & \mathrm{Comod}_{X_+} \arrow[l, shift left=2, "-\star X"] \end{tikzcd},$$

where R_X is equipped with the model category structure inherited from $(\mathrm{sSet})_{\mathcal{E}}$.

Advantage: know sets of generating (acyclic) cofibrations for the model category structure on Comod_{X_+}

Useful notation

Notation

Let f and g be morphisms in a category C*. If for every commutative diagram in* C

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "b"'] \arrow[ur, dashed, "c"] & \cdot \end{tikzcd}$$

the dotted lift c exists, then we write $f \boxslash g$.

If $\mathcal{X}$ is a class of morphisms C*, then*

$$\mathcal{X}^{\boxslash} = \{f \in \mathrm{Mor}\, C \mid x \boxslash f \quad \forall x \in \mathcal{X}\}.$$

Definition of left-induced structures

Definition

Let $\mathsf{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathsf{M}$ *be an adjoint pair of functors, where* $(\mathsf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ *is a model category, and* C *is a bicomplete category. If the triple of classes of morphisms in* C

$$\Big(\big(U^{-1}(\mathcal{C} \cap \mathcal{W})\big)^{\boxslash}, U^{-1}(\mathcal{C}), U^{-1}(\mathcal{W}) \Big)$$

satisfies the axioms of a model category, then it is a *left-induced model structure* *on* C.

Existence of left-induced structures

Theorem
(Bayeh-H.-Karpova-Kędziorek-Riehl-Shipley, 2014)

Let $\mathsf{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathsf{M}$ *be an adjoint pair of functors, where* C *is locally presentable, and* $(\mathsf{M}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ *is a combinatorial model category.*
If

$$(U^{-1}\mathcal{C})^{\boxslash} \subset U^{-1}\mathcal{W},$$

then the left-induced model structure on C *exists and is cofibrantly generated.*

Application to Comod_{X_+}

Applying the existence theorem from the previous slide, we get...

Theorem (H.-Shipley, 2014)

There is a model category structure $(\mathsf{Comod}_{X_+})_{\mathcal{E}}$ *left-induced from* $(\mathsf{sSet}_*)_{\mathcal{E}}$ *by the adjunction*

$$\begin{tikzcd} \mathsf{Comod}_{X_+} \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{sSet}_* . \arrow[l, shift left, "F_{X_+}"] \end{tikzcd}$$

Moreover if (X, μ, x_0) *is a simplicial monoid, then*

$$\big((\mathsf{Comod}_{X_+})_{\mathcal{E}}, \otimes, (S^0, \rho_u)\big)$$

is a monoidal model category satisfying the monoid axiom.

Advantages to this approach become clear when we stabilize.

The non simply connected case

Theorem

Let $q : \widetilde{X} \to X$ be a universal cover. The adjunctions

$$\mathsf{R}_{\widetilde{X}} \underset{q^*}{\overset{q_*}{\rightleftarrows}} \mathsf{R}_X \ , \quad \mathsf{Comod}_{\widetilde{X}_+} \underset{q^*}{\overset{q_*}{\rightleftarrows}} \mathsf{Comod}_{X_+}$$

right-induce left proper, cofibrantly generated model category structures $(\mathsf{R}_X)_{\mathcal{H}q^*}$ and $(\mathsf{Comod}_{X_+})_{\mathcal{H}q^*}$ from $(\mathsf{R}_{\widetilde{X}})_{\mathcal{H}\mathbb{Z}}$ and $(\mathsf{Comod}_{\widetilde{X}})_{\mathcal{H}\mathbb{Z}}$. In particular, the adjunction

$$(\mathsf{R}_X)_{\mathcal{H}q^*} \underset{-\star X}{\overset{-/X}{\rightleftarrows}} (\mathsf{Comod}_{X_+})_{\mathcal{H}q^*}$$

is a Quillen equivalence.

Koszul duality

Theorem

If X is a reduced simplicial set, and $\mathcal{E}$ is any generalized homology theory, then there is a Quillen equivalence

$$\begin{tikzcd}
(\mathrm{Mod}_{\mathbb{G}X})_{\mathcal{E}} \arrow[r, shift left, "- \wedge_{(\mathbb{G}X)_+} (\mathbb{P}X)_+"] \arrow[r, phantom, "\perp"] & (\mathrm{Comod}_{X_+})_{\mathcal{E}} \arrow[l, shift left]
\end{tikzcd}.$$

The stable case

Theorem (H.-Shipley, 2014)

There are combinatorial, left proper, spectral model category structures

$$\mathrm{Sp}_{\mathcal{E}}, \quad (\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{st}}_{\mathcal{E}}, \quad \textit{and} \quad (\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{left}}_{\mathcal{E}},$$

where the first two are stabilized from $(\mathrm{sSet}_*)_{\mathcal{E}}$ *and* $(\mathrm{Comod}_{X_+})_{\mathcal{E}}$*, and the third is left-induced from the first.*
In particular, the functors

$$(\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{st}}_{\mathcal{E}} \xrightarrow{\mathrm{Id}} (\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{left}}_{\mathcal{E}} \xrightarrow{U} \mathrm{Sp}_{\mathcal{E}}$$

are left Quillen, and weak equivalences and fibrations in $(\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{left}}_{\mathcal{E}}$ *are created by* U*.*

REMARKS

- The description of $(\mathrm{Comod}_{X_+})_{\mathcal{E}}$ as a left-induced structure is crucial for this proof.
- $(\mathrm{Comod}_{\Sigma^\infty X_+})^{?}_{\mathcal{H}\mathbb{Z}} = (\mathrm{Comod}_{\Sigma^\infty X_+})^{?}_{\pi^s_*}$ for $? =$ st or left
- If X is a simplicial monoid, then $(\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{left}}_{\mathcal{E}}$ admits a monoidal structure satisfying the monoid axiom.
- Koszul duality stabilizes: if X is a reduced simplicial set and $\mathcal{E}$ any generalized homology theory, then there is a Quillen equivalence

$$\begin{tikzcd} (\mathrm{Mod}_{\Sigma^\infty(\mathbb{G}X)_+})^{\mathrm{st}}_{\mathcal{E}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{st}}_{\mathcal{E}} \arrow[l, shift left] \end{tikzcd},$$

Algebraic homotopy of comodules

If H is a simplicial monoid, then

$$\mathsf{Alg}_{\Sigma^\infty H_+} = \text{the category of monoids in } \mathsf{Comod}_{\Sigma^\infty H_+}.$$

Objects: symmetric ring spectra $\mathbf{R}$ endowed with a coassociative, counital morphism

$$\rho : \mathbf{R} \to \mathbf{R} \wedge \Sigma^\infty H_+$$

of symmetric ring spectra.

Corollary

There is a cofibrantly generated model category structure $(\mathsf{Alg}_{\Sigma^\infty H_+})_\mathcal{E}$ with respect to which the forgetful/cofree adjunction

$$\begin{tikzcd} (\mathsf{Alg}_{\Sigma^\infty H_+})_\mathcal{E} \arrow[r, shift left=2, "U"] \arrow[r, phantom, "\perp"] & (\mathsf{Alg})_\mathcal{E} \arrow[l, shift left=2, "-\wedge\Sigma^\infty H_+"] \end{tikzcd}$$

is a Quillen pair.

Consequence for Hopf-Galois theory

Can now formulate rigorously the notion of the homotopy coinvariants of the $\Sigma^\infty H_+$-coaction on an object $(\mathbf{R}, \rho)$ in $\mathrm{Alg}_{\Sigma^\infty H_+}$, which is essential in the definition of a homotopic Hopf-Galois extension [Rognes].

Definition

A model for the homotopy coinvariants of $(\mathbf{R}, \rho)$ is the equalizer in $\mathrm{Alg}_{\Sigma^\infty H_+}$

$$(\mathbf{R}, \rho)^{hco\,\Sigma^\infty H_+} = \mathrm{equal}\left(\mathbf{R}^f \underset{\mathbf{R}^f \wedge \eta}{\overset{\rho^f}{\rightrightarrows}} \mathbf{R}^f \wedge \Sigma^\infty H_+\right),$$

where $(\mathbf{R}^f, \rho^f)$ is a fibrant replacement for $(\mathbf{R}, \rho)$ in $\mathrm{Alg}_{\Sigma^\infty H_+}$, and $\eta : \mathbf{S} \to \Sigma^\infty H_+$ is the unit of the ring spectrum $\Sigma^\infty H_+$.

Consequences for K-theory

From comodules to K-theory

Notation

$A(X; \mathcal{E}_*)$ denotes the K-theory of $\mathrm{R}_X^{\mathrm{hf}}$ with the usual cofibrations, but with $\mathcal{E}_*$-equivalences as weak equivalences.

Theorem (H.-Shipley, 2014)

For any simplicial set X and any generalized reduced homology theory $\mathcal{E}_*$, $(\mathrm{Comod}_{X_+})_{\mathcal{E}}^{\mathrm{hf}}$ is a Waldhausen category, and there are natural weak equivalences of K-theory spectra

$$A(X; \mathcal{E}_*) \xrightarrow{\simeq} K\big((\mathrm{Comod}_{X_+})_{\mathcal{E}}^{\mathrm{hf}}\big) \xleftarrow{\simeq} K\big((\mathrm{Mod}_{\Omega X})_{\mathcal{E}}^{\mathrm{hf}}\big).$$

The non simply connected case

Let $q : \widetilde{X} \to X$ be a universal cover. Recall $(\mathrm{R}_X)_{\mathcal{H}q^*}$ and $(\mathrm{Comod}_{X_+})_{\mathcal{H}q^*}$. Note that $\mathcal{H}q^* = \mathcal{H}\mathbb{Z}_*$ if X is simply connected.

Lemma

$A(X) \xrightarrow{\simeq} A(X; \mathcal{H}q^*)$ is a weak equivalence for every connected simplicial set X.

Corollary

There is a natural weak equivalence of K-theory spectra

$$A(X) \xrightarrow{\simeq} K\big((\mathrm{Comod}_{X_+})^{\mathrm{hf}}_{\mathcal{H}q^*}\big).$$

In particular, if X is simply connected, then

$$A(X) \xrightarrow{\simeq} K\big((\mathrm{Comod}_{X_+})^{\mathrm{hf}}_{\mathcal{H}\mathbb{Z}}\big).$$

The stable version

Corollary

There is a natural weak equivalence of K-theory spectra

$$A(X) \xrightarrow{\simeq} K\big((\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{hf}}\big),$$

where $(\mathrm{Comod}_{\Sigma^\infty X_+})^{\mathrm{hf}}$ denotes the category of homotopically finite comodules over $\Sigma^\infty X_+$, with cofibrations and weak equivalences inherited from the stabilization of the model structure on $(\mathrm{Comod}_{X_+})_{\mathcal{H}q^*}$.

Happy Birthday, Tom!