J. K-Theory **6** (2010), 241–261
doi:10.1017/is009009019jkt093

Generalized orientations and the Bloch invariant

by

MICHEL MATTHEY, WOLFGANG PITSCH AND JÉRÔME SCHERER*

Abstract

For compact hyperbolic 3-manifolds we lift the Bloch invariant defined by Neumann and Yang to an integral class in $K_3(\mathbb{C})$. Applying the Borel and the Bloch regulators, one gets back the volume and the Chern-Simons invariant of the manifold. We perform our constructions in stable homotopy theory, pushing a generalized orientation of the manifold directly into K-theory. On the way we give a purely homotopical construction of the Bloch-Wigner exact sequence which allows us to explain the $\mathbb{Q}/\mathbb{Z}$ ambiguity that appears in the non-compact case.

Key Words: Hyperbolic manifold, generalized orientation, scissors congruence, hyperbolic volume, Bloch invariant

Mathematics Subject Classification 2000: Primary 57M27; Secondary 19E99, 55N20, 55P43, 57M50

Introduction

Suppose that Γ is a discrete group such that the classifying space $B\Gamma$ has a model which is a closed orientable smooth manifold M of dimension m. Here as usual closed means compact and without boundary. According to the *Borel conjecture* for Γ, the homeomorphism type of M should be completely determined by the isomorphism type of Γ. Therefore the question arises of how much of the smooth geometry of M is encoded in the group Γ. Similarly, recall that by the celebrated *Mostow Rigidity*, if M is a closed connected orientable hyperbolic manifold of dimension $n \geq 3$, then not only the Borel conjecture holds for Γ, but the isometry type of M is also completely determined by Γ. So, in this case, the question refines to how the metric geometry of M, typically the *hyperbolic volume* $\mathrm{vol}(M)$ or the *Chern-Simons invariant* $\mathrm{CS}(M)$, can be recovered from Γ.

Such questions have been addressed for instance by Goncharov [Gon99], and Neumann and Yang [NY99]. In the three dimensional case, they obtained respectively a rational algebraic K-theoretical invariant, and a *Bloch invariant* in the

*The second and third authors are supported by the program Ramón y Cajal, MEC, Spain and by FEDER/MEC grant MTM2007-61545. The third author was partially supported by the Mittag-Leffler Institute in Sweden and the Max-Planck-Institut für Mathematik in Germany.

Bloch group which is in the scissors congruence group of hyperbolic 3-space $\mathcal{P}(\mathbb{C})$. The Bloch group is naturally a sub-quotient of $K_3(\mathbb{C})$ and it is therefore natural to try to lift the invariant to the latter group. A first attempt can be found in Cisneros-Molina and Jones [CMJ03] where they revisited Neumann and Yang's work from a homotopical perspective. From the point of view of scissors-congruence theory, lifts to extended Bloch groups are considered in [Neu04], and in the following articles [DZ06] and [GZ07]. One of the objectives is, using tools from complex analysis on Riemann surfaces, to get formulas leading to explicit computations. In particular in [GZ07] the authors obtain a combinatorial description of the group $H_3(SL_2(\mathbb{C});\mathbb{Z}))$. Our approach on the other side is purely homotopical.

There is one constant in all three approaches: the invariant is obtained basically by pushing a fundamental class in ordinary homology into $\mathcal{P}(\mathbb{C})$. The main tool to relate $\mathcal{P}(\mathbb{C})$ to K-theory is the Bloch-Wigner exact sequence first published by Dupont-Sah [DS82] and by Suslin [Sus90]. One gets directly a class in the homology of $SL_2\mathbb{C}$ by considering a Spin-structure on the hyperbolic manifold, so, to define the invariant in K-theory one has to lift this fundamental class through a Hurewicz homomorphism, which leads to an ambiguity in the construction. One of the motivations of this work is to shed some light on the origin of this ambiguity and on the possibilities to reduce it. In [Gon99] for instance, it is removed by using rational coefficients.

Our starting point is the observation that the classical Bloch-Wigner exact sequence is a part of the long exact sequence in stable homotopy of a cofibration. Thus instead of a Spin-structure, which yields a KO-orientation [ABS64], we are lead to consider an orientation in stable homotopy theory, and this is provided geometrically by a stable parallelization of the (hyperbolic) 3-manifold. The first advantage of this point of view is that it gives directly a class in K-theory. It is also possible to discuss the influence of the various choices that are involved in the choice of a parallelization (Spin structure, p_1-structure) on the final invariant. In particular we show that the $\mathbb{Q}/\mathbb{Z}$-ambiguity that appears in the non-compact case is irreducible from a purely homotopical point of view.

For compact manifolds our main result is:

Theorem A. *Let M be a closed oriented hyperbolic manifold of dimension* 3 *with fundamental group $\Gamma = \pi_1(M)$. Then, to any stable parallelization of the tangent bundle of M corresponds, in a canonical way, a K-theory class $\gamma(M) \in K_3(\mathbb{C})$, which depends only and effectively on the underlying Spin-structure.*

There are two *regulators* defined on $K_3(\mathbb{C})$, the Borel regulator and the Bloch regulator. The insight of Goncharov and Neumann-Yang tells us that their values on the invariant gives back the volume and the Chern-Simons invariant of the manifold.

Corollary.(Neumann-Yang, [NY99, Theorem 1.3]) *The hyperbolic volume of M is determined by the equality*

$$\text{bo-reg}(\gamma(M)) = \frac{\mathrm{vol}(M)}{2\pi^2}$$

of real numbers, where bo-reg: $K_3(\mathbb{C}) \to \mathbb{R}$ *is the Borel regulator for the field of complex numbers* $\mathbb{C}$. *Furthermore, for the Chern-Simons invariant* $\mathrm{CS}(M)$ *of* M *we have the congruence*

$$\mu(\gamma(M)) \equiv \frac{-CS(M) + i \cdot \mathrm{vol}(M)}{2\pi^2} \pmod{\mathbb{Q}}$$

of complex numbers. Here μ *stands for the composite*

$$\mu\colon K_3(\mathbb{C}) \xrightarrow{\text{bw}} \mathcal{B}(\mathbb{C}) \xrightarrow{\text{bl-reg}} \mathbb{C}/\mathbb{Q},$$

where bw *is the Bloch-Wigner map for the field* $\mathbb{C}$, $\mathcal{B}(\mathbb{C})$ *is the Bloch group of* $\mathbb{C}$, *and* bl-reg *is the Bloch regulator for* $\mathbb{C}$.

In the non-compact case, the problem is more intricate. The main problem is that one has to start with a fundamental class in a relative (generalized) homology group, and this yields naturally a relative orientation class. Even if we do not have to invert a Hurewicz homomorphism we still end up with a $\mathbb{Q}/\mathbb{Z}$ ambiguity, compare with [CMJ03, Remark 8.9].

Theorem B. *Let* M *be a non-compact oriented hyperbolic manifold of dimension* 3 *with finite volume. Let* $\Gamma = \pi_1(M)$ *be its fundamental group. Then, to any stable parallelization of the tangent bundle of* M *correspond* $\mathbb{Q}/\mathbb{Z}$ *natural* K*-theory classes* $\gamma(M)$ *in* $K_3(\mathbb{C})$, *which depend only on the underlying Spin-structure.*

Again, as in the compact case, this class "computes" the volume, as previously shown by Neumann and Yang.

The approach via orientations in generalized homology theories allows to extend the construction to K-theory groups that retain a-priori more information on the fundamental group Γ than $K_3(\mathbb{C})$. Indeed, our original plan was to construct an invariant in the algebraic K-theory of the group ring $\mathbb{Z}\Gamma$. The fact that the Bloch-Wigner exact sequence can be reformulated in stable homotopy simplified the construction. However, we decided to include our original construction in Appendix A for two reasons: the intimate relation of $K\mathbb{Z}_*(\mathbb{Z}\Gamma)$ with the Isomorphism Conjectures, [FJ93], and because it might lead to explicit computations.

The plan of the article is the following. Section 1 is a short reminder on the theory of orientations of manifolds. Section 2 is devoted to the Bloch-Wigner exact sequence. Theorem A and its corollary are proved in Section 3. The non-compact case, Theorem B, is the object of Section 4. Finally Appendix A contains the construction of an invariant in $K_3(\mathbb{Z}\Gamma)$. In this article all groups are discrete and we freely identify the homology of the discrete group G with that of its classifying space BG.

We started this project in February 2005, but the paper was completed only after the first author's death. It is dedicated to the memory of our friend Michel Matthey.

Acknowledgements. We would like to thank Joan Porti, José Burgos, and Johan Dupont for enlightening discussions. We also would like to thank the referee and the editors of the Journal of K-Theory, in particular Max Karoubi; their remarks and questions helped us to clarify a number of points in the text.

1. Parallelizations and orientations

Let M be a closed compact connected smooth manifold of dimension d. We explain in this section the relationship between stable parallelizations of the tangent bundle of M and orientations of M with respect to the sphere spectrum $\mathbb{S}$. For manifolds there are two ways to view orientations. The first one, arising from orientations of vector bundles, is cohomological in essence and the second one, arising from patching local compatible orientations, is homological in essence. Both definitions agree via the so-called S-duality. We call a manifold *orientable* if it is so in the classical sense (i.e. with respect to the Eilenberg-McLane spectrum $H\mathbb{Z}$). In this section E denotes a ring spectrum with unit $\varepsilon : \mathbb{S} \to E$.

1.1. Cohomological definition

Let ν_M be the stable normal bundle of M and $Th(\nu_M)$ its Thom spectrum. For each $m \in M$, consider the map from the Thom spectrum of this point induced by the inclusion $j_m : \mathbb{S} \to Th(\nu_M)$. An *$E$-orientation* of M is a class $t \in E^0(Th(\nu_M))$ such that for some (and hence every) point $m \in M$ $j_m^*(t) = \pm\varepsilon \in \pi_0(E) \cong E^0(\mathbb{S})$.

A particularly convenient setting is when the manifold is stably parallelizable, i.e. its normal bundle is stably trivial (and hence its tangent bundle also). A given parallelization ι provides a trivialization of the Thom spectrum of the normal bundle of M:

$$DT(\iota) : Th(\nu_M) \xrightarrow{\simeq} \Sigma^\infty M_+.$$

By collapsing M to a point we obtain hence a map $Th(\nu_M) \to \mathbb{S}$ to the sphere spectrum representing a cohomology class in $\mathbb{S}^0(Th(\nu_M))$. Composing with the unit $\varepsilon : \mathbb{S} \to E$ we get an E-orientation.

Example 1.1 Recall Stiefel's result that any orientable 3-manifold is parallelizable (see [MS74, Problem 12-B]) i.e., the tangent bundle $\tau : M \to BO(3)$ is trivial. As these trivializations correspond to lifts of the map τ to the universal cover $EO(3)$ up to homotopy, one can apply obstruction theory to count them. Lifts to the 1-skeleton correspond to classical orientations and there are $H^0(M;\mathbb{Z}/2\mathbb{Z})$ possible choices. Further lifts to the 2-skeleton correspond to Spin-structures, and there are $H^1(M;\mathbb{Z}/2\mathbb{Z})$ choices at this stage. Finally, to lift further across the 3-skeleton one gets $H^3(M;\mathbb{Z})$ choices, the so called p_1-structures, where p_1 stands for the first Pontrjagin class.

1.2. Homological definition

A *fundamental class* for M with respect to the homology theory E is an element $t \in E_d(M)$ such that for some (and therefore every) point $m \in M$ the image of t in $E_d(M, M-m) \simeq \tilde{E}_d(S^d) \simeq \tilde{E}_0(S^0) = \pi_0(E)$ is $\pm\varepsilon$. Notice in particular that the unit $\varepsilon : \mathbb{S} \to E$ canonically provides fundamental classes for all spheres S^d.

Example 1.2 Consider the sphere spectrum $\mathbb{S}$. Then the corresponding reduced homology theory is stable homotopy, $\tilde{\mathbb{S}}_n(X) \cong \pi_n^s(X)$. An $\mathbb{S}$-orientation for M is thus an element in $\mathbb{S}_d(M)$ with the property that its image in $\mathbb{S}_d^s(M, M-m) \cong \pi_d^s(S^d) \cong \mathbb{Z}$ is a generator.

1.3. S-duality

We now turn to the connection between the homological and cohomological point of view. We follow Rudyak's treatment of S-duality, [Rud98], see also Switzer [Swi02] or Adams [Ada74].

Definition 1.3 Let A, A^* be two spectra. A *duality morphism* or *duality* between A and A^* is a map of spectra $u : \mathbb{S} \to A \wedge A^*$ such that for every spectrum E the following homomorphisms are isomorphisms :

$$\begin{array}{rcl} u_E : [A, E] & \longrightarrow & [\mathbb{S}, E \wedge A^*] \\ \phi & \longmapsto & (\phi \wedge 1_{A^*}) \circ u \end{array}$$

$$\begin{array}{rcl} u^E : [A^*, E] & \longrightarrow & [\mathbb{S}, A \wedge E] \\ \phi & \longmapsto & (1_A \wedge \phi) \circ u \end{array}$$

The spectra A and A^* are said to be *S-dual*. Two spectra A and B are called *n-dual*, where $n \in \mathbb{Z}$, if A and $\Sigma^n B$ are S-dual.

Definition 1.4 Fixing two duality maps $u : \mathbb{S} \to A \wedge A^*$ and $v : \mathbb{S} \to B \wedge B^*$, the S-*dual* of a map $f : A \to B$ is the image $f^* : B^* \to A^*$ of f under the isomorphism:

$$D : [A,B] \xrightarrow{u_B} [\mathbb{S}, B \wedge A^*] \xrightarrow{(v^{A^*})^{-1}} [B^*, A^*].$$

In particular $f \in [A,B]$ is S-dual to $g \in [B^*, A^*]$ if and only if $u_B(f) = v^{A^*}(g)$.

Example 1.5 For any integer n the spectra $\mathbb{S}^n$ and $\mathbb{S}^{-n}$ are S-dual. The duality map is simply the canonical equivalence $\mathbb{S} \to \mathbb{S}^n \wedge \mathbb{S}^{-n}$.

1.4. Orientations and S-duality for manifolds

For closed manifolds S-duality was defined by Milnor and Spanier in [MS60]. As we will need the precise form of the duality map we give it in detail. Choose an embedding $M \hookrightarrow S^N$ into a high-dimensional sphere and let U be a tubular neighbourhood of M. The open manifold U can be viewed as the total space of the normal disc bundle of M, and the quotient $\overline{U}/\partial U$ is therefore a Thom space for the normal bundle. Denote by $p : \overline{U} \to M$ the projection and by $\Delta : \overline{U} \to \overline{U} \times M$ the map $\Delta(a) = (a, p(a))$. Then Δ induces a map $\Delta' : \overline{U}/\partial U \to \overline{U}/\partial U \wedge M_+$. Denote by $C : S^N \to \overline{U}/\partial U$ the map induced by collapsing the complement of U into a point. Then we have a map $f\ : S^N \xrightarrow{C} \overline{U}/\partial U \xrightarrow{\Delta'} (\overline{U}/\partial U) \wedge M_+$. The duality morphism is then

$$u = \Sigma^{-N}\Sigma^\infty f : \mathbb{S} \to Th\nu_M \wedge \Sigma^{-d}\Sigma^\infty M_+.$$

It induces the duality bijection $u_E : [Th(\nu_M), E] \to [\mathbb{S}, E \wedge \Sigma^{-d}\Sigma^\infty M_+]$ for any spectrum E.

Theorem 1.6 *[Rud98, Corollary V.2.6] Let M be a closed E-orientable manifold. The duality map u_E yields a bijective correspondence between cohomological orientations of M and fundamental classes of M with respect to E.* □

1.5. The case of 3-manifolds

In Example 1.1 we have seen that 3-manifolds are orientable in the cohomological sense. Therefore by Theorem 1.6 they admit fundamental classes. We describe now the relationship between parallelizations and homological orientations for 3-manifolds. Since we counted the former in Example 1.1 we will first count the latter.

Lemma 1.7 *Let M be an orientable closed manifold of dimension 3. The Atiyah-Hirzebruch spectral sequence for the stable homotopy of M collapses at E^2.*

Proof: The spectral sequence is concentrated on the first four columns of the first quadrant. The first column $H_0(M;\mathbb{S}_q) \cong \pi_q^s$ always survives to E^∞ since a point is a retract of M. Since M is $\mathbb{S}$-orientable, the suspension spectrum of the 3-sphere is a retract of $\Sigma^\infty M$, so that the fourth column $H_3(M;\mathbb{S}_q) \cong \pi_q^s$ also survives. Therefore all differentials must be zero. □

Proposition 1.8 *Let M be an orientable closed 3-manifold. Fundamental classes of M with respect to $\mathbb{S}$ are parametrized by $\pi_3^s \times H_1(M;\mathbb{Z}/2\mathbb{Z}) \times H_2(M;\mathbb{Z}/2\mathbb{Z}) \times \{\pm 1\}$.*

Proof: This follows from the previous lemma since the homomorphism $\mathbb{S}_3(M) \to \mathbb{S}_3(M, M-m)$ can be identified with the edge homomorphism $\mathbb{S}_3(M) \to H_3(M;\mathbb{Z})$. Fixing an orientation tells us that the image of t must be a fixed generator of $H_3(M;\mathbb{Z})$. □

Example 1.9 There are precisely $2 \cdot |\pi_3^s| = 48$ different orientations of the sphere S^3 with respect to stable homotopy.

If an $\mathbb{S}$-orientation of M is given, a change of trivialization can be used to modify the class in $\mathbb{S}_3(M)$ via the Dold-Thom isomorphisms:

$$\mathbb{S}_3(M) \xrightarrow{DT(\iota)^{-1}} \mathbb{S}_3(Th(\nu_M)) \xrightarrow{DT(\iota')} \mathbb{S}_3(M).$$

Lemma 1.10 *Given two stable parallelizations of S^3 which differ only by a p_1-structure $\alpha \in H^3(S^3;\mathbb{Z})$, the corresponding $\mathbb{S}$-orientations differ then by $J\alpha$, where $J : \mathbb{Z} \cong \pi_3 SO \twoheadrightarrow \pi_3^s \cong \mathbb{Z}/24$ is the stable J-homomorphism.*

Proof: The change of trivialization is controlled by a map between total spaces of trivial bundles $S^3 \times \mathbb{R}^N \to S^3 \times \mathbb{R}^N$, for some large integer N. At the level of Thom spaces we get a homotopy equivalence $f : S^{N+3} \vee S^N \to S^{N+3} \vee S^N$. Fix the canonical $\mathbb{S}$-orientation t corresponding to the wedge summand inclusion $S^{N+3} \to S^{N+3} \vee S^N$ in $\pi_{N+3}(S^{N+3} \vee S^N) \cong \pi_3^s(S^3_+) \cong \mathbb{S}_3(S^3)$ and modify it by f. The edge homomorphism $e : \mathbb{S}_3(S^3) \to \pi_3^s(S^3)$ takes both t and ft to 1, and the corresponding element in $\operatorname{Ker} e$ is given by the map

$$S^{N+3} \xrightarrow{i_1} S^{N+3} \vee S^N \xrightarrow{f} S^{N+3} \vee S^N \xrightarrow{p_2} S^N.$$

This map is determined by its homotopy cofiber, a two cell complex which is seen to be homotopy equivalent to $S^N \cup_{J\alpha} e^{N+4}$, see [Ada66, Lemma 10.1]. We conclude then since J is an epimorphism in dimension 3, [Ada66, Theorem 1.5]. □

Proposition 1.11 *Let M be an oriented, closed 3-manifold. The $\mathbb{S}$-orientations of M obtained from the stable parallelizations may differ by an arbitrary element of $\mathbb{Z}/24 \cong \pi_3^s \subset \operatorname{Ker} e$.*

Proof: One obtains both stable parallelizations and $\mathbb{S}$-orientations for S^3 from the ones for M by collapsing the 2-skeleton. □

2. The Bloch-Wigner exact sequence

In this section we identify the Bloch-Wigner exact sequence with an exact sequence in stable homotopy whereas the classical point of view is homological.

2.1. Scissors congruence group of hyperbolic 3*-space*

A standard reference for this section is Dupont-Sah [DS82], see also Dupont [Dup01] or Suslin [Sus90]. Denote by $\mathrm{Isom}^+(\mathcal{H}^3)$ the group of orientation-preserving isometries of the hyperbolic 3-space $\mathcal{H}^3$.

Definition 2.1 The *scissors congruence group* $\mathcal{P}(\mathcal{H}^3)$ is the free abelian group of symbols $[P]$ for all polytopes P in $\mathcal{H}^3$, modulo the relations:

1. $[P]-[P']-[P'']$ if $P = P' \cup P''$ and $P' \cap P''$ has no interior points;

2. $[gP]-[P]$ for $g \in \mathrm{Isom}^+(\mathcal{H}^3)$.

One defines analogously $\mathcal{P}(\overline{\mathcal{H}}^3)$ where one allows some vertices of the polytopes to be ideal points and $\mathcal{P}(\partial\mathcal{H}^3)$ where the polytopes are all ideal polytopes (actually there is a subtlety with the latter group, see [Dup01, Chapter 8]). Finally there is a more algebraic description of these groups.

Definition 2.2 Let $\mathcal{P}(\mathbb{C})$ denote the abelian group generated by $z \in \mathbb{C}-\{0,1\}$ and satisfying, for $z_1 \neq z_2$, the relations:

$$z_1 - z_2 + \frac{z_2}{z_1} - \frac{1-z_2}{1-z_1} + \frac{1-z_2^{-1}}{1-z_1^{-1}}.$$

The four groups are related by:

Theorem 2.3 *[Dup01, Corollary 8.18] The natural inclusions induce isomorphisms*

$$\mathcal{P}(\mathcal{H}^3) \cong \mathcal{P}(\overline{\mathcal{H}}^3) \cong \mathcal{P}(\partial\mathcal{H}^3).$$

Moreover these groups are isomorphic to $\mathcal{P}(\mathbb{C})^-$, the (-1)-eigenspace of $\mathcal{P}(\mathbb{C})$ for complex conjugation. □

2.2. The Bloch-Wigner exact sequence

Recall that the group $\mathrm{Isom}^+(\mathcal{H}^3)$ is isomorphic to $PSL_2\mathbb{C} = SL_2\mathbb{C}/\{\pm Id\}$. It acts naturally on the boundary of hyperbolic 3-space. Fix a point $x \in \partial\mathcal{H}^3$ and denote by $Bo \subset SL_2\mathbb{C}$ the preimage of the stabilizer of x. As a group, Bo is isomorphic to the semi-direct product $\mathbb{C} \rtimes \mathbb{C}^*$, where $z \in \mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication by z^2. These groups are all considered only as discrete groups. Let us then denote by $Cof(i_{Bo})$ the homotopy cofibre of the map $i_{Bo} : BBo \to BSL_2\mathbb{C}$. The following is an integral analogue of [Gon99, Lemma 2.14].

Lemma 2.4 *For $n \geq 1$ we have a commutative diagram where the vertical arrows are the Hurewicz homomorphisms and the horizontal arrows are induced by the projections $\mathbb{C} \rtimes \mathbb{C}^* \twoheadrightarrow \mathbb{C}^*$:*

$$\begin{tikzcd}
\pi_n^s(B(\mathbb{C} \rtimes \mathbb{C}^*)) \arrow[r] \arrow[d, "\wr"] & \pi_n^s(B\mathbb{C}^*) \arrow[d] \\
H_n(B(\mathbb{C} \rtimes \mathbb{C}^*);\mathbb{Z}) \arrow[r, "\sim"] & H_n(B\mathbb{C}^*;\mathbb{Z}).
\end{tikzcd}$$

Proof: From the exact sequence of groups $1 \to \mathbb{C} \to \mathbb{C} \rtimes \mathbb{C}^* \to \mathbb{C}^* \to 1$ we get a fibration $B\mathbb{C} \to B(\mathbb{C} \rtimes \mathbb{C}^*) \to B\mathbb{C}^*$. We will prove that the Atiyah-Hirzebruch spectral sequence for stable homotopy

$$H_p(B\mathbb{C}^*;\pi_q^s(B\mathbb{C})) \Rightarrow \pi_{p+q}^s(B(\mathbb{C} \rtimes \mathbb{C}^*))$$

collapses. Since the stable stems π_n^s are torsion groups in degree $n \geq 1$ and $\mathbb{C}$ is a rational vector space, the Hurewicz homomorphism $\pi_n^s(B\mathbb{C}) \to H_n(B\mathbb{C};\mathbb{Z})$ is an isomorphism, which identifies this spectral sequence with the ordinary homological spectral sequence. In particular the map $\pi_n^s(B(\mathbb{C} \rtimes \mathbb{C}^*)) \to H_n(B(\mathbb{C} \rtimes \mathbb{C}^*);\mathbb{Z})$ is an isomorphism.

The idea of the second part of the proof goes back at least to Suslin's [Sus84, Corollary 1.8]. An element $n \in \mathbb{C}^*$ acts by multiplication by n^2 on $\mathbb{C}$, hence by multiplication by n^{2q} on $H_q(B\mathbb{C};\mathbb{Z}) \cong \Lambda^q\mathbb{C}$ for any $q \geq 1$. The map induced by conjugation in a group G by an element g together with the action of the same g on a G-module M induces the identity in homology with coefficients in M. As $\mathbb{C}^*$ is abelian, in our case we have that multiplication by n^{2q} is the identity on $H_p(B\mathbb{C}^*;H_q(B\mathbb{C};\mathbb{Z}))$. But multiplication by $n^{2q} - 1$ is an isomorphism on the $\mathbb{C}^*$-module $H_q(B\mathbb{C};\mathbb{Z})$, so that $H_p(B\mathbb{C}^*;H_q(B\mathbb{C};\mathbb{Z})) = 0$ for $q \geq 1$ and therefore $H_*(B(\mathbb{C} \rtimes \mathbb{C}^*);\mathbb{Z}) \cong H_*(B\mathbb{C}^*;\mathbb{Z})$. □

Lemma 2.5 *For $n \leq 3$, the Hurewicz homomorphism $\pi_n^s(BSL_2\mathbb{C}) \to \widetilde{H}_n(SL_2\mathbb{C};\mathbb{Z})$ is an isomorphism.*

Proof: The group $SL_2\mathbb{C}$ is perfect and $\widetilde{H}_2(SL_2\mathbb{C};\mathbb{Z})$ is a rational vector space [Dup01, Corollary 8.20]. The result now follows from an easy Atiyah-Hirzebruch spectral sequence argument. $\square$

Proposition 2.6 *There is a commutative diagram with vertical isomorphisms and exact rows*

$$\begin{tikzcd}
\mathbb{Q}/\mathbb{Z} \arrow[r, hook] \arrow[d] & \pi_3^s(BSL_2\mathbb{C}) \arrow[r] \arrow[d] & \pi_3^s(Cof(i_{Bo})) \arrow[r] \arrow[d] & \pi_2^s(BBo) \arrow[r, two heads] \arrow[d] & \pi_2^s(BSL_2\mathbb{C}) \arrow[d] \\
\mathbb{Q}/\mathbb{Z} \arrow[r, hook] & H_3(SL_2\mathbb{C};\mathbb{Z}) \arrow[r] & \mathcal{P}(\mathbb{C}) \arrow[r] & \Lambda^2(\mathbb{C}^*/\mu_{\mathbb{C}}) \arrow[r, two heads] & H_2(SL_2\mathbb{C};\mathbb{Z})
\end{tikzcd}$$

where the bottom row is the Bloch-Wigner exact sequence.

Proof: The stable Hurewicz homomorphism permits us to compare the long exact sequences of the cofibration $BBo \to BSL_2\mathbb{C} \to Cof(i_{Bo})$ in stable homotopy and in ordinary homology:

$$\begin{tikzcd}
\pi_3^s(BBo) \arrow[r] \arrow[d, "\cong"] & \pi_3^s(BSL_2\mathbb{C}) \arrow[r] \arrow[d, "\cong"] & \pi_3^s(Cof(i_{Bo})) \arrow[r] \arrow[d] & \pi_2^s(BBo) \arrow[r] \arrow[d, "\cong"] & \pi_2^s(BSL_2\mathbb{C}) \arrow[d, "\cong"] \\
H_3(\mathbb{C}^*;\mathbb{Z}) \arrow[r] & H_3(SL_2\mathbb{C};\mathbb{Z}) \arrow[r] & H_3(Cof(i_{Bo});\mathbb{Z}) \arrow[r] & H_2(\mathbb{C}^*;\mathbb{Z}) \arrow[r] & H_2(SL_2\mathbb{C};\mathbb{Z})
\end{tikzcd}$$

The marked isomorphisms are given by Lemmas 2.4 and 2.5. It remains thus to compare the bottom exact sequence with the Bloch-Wigner exact sequence. We have to return to its computation by Suslin, [Sus90].

Let P_* be a projective resolution of $\mathbb{Z}$ over $SL_2\mathbb{C}$ and consider the complex C_* of $(n+1)$-uples of distinct points in $\partial\mathcal{H}^3$, [Dup01, Chapter 2]. The naturally augmented complex $\epsilon : C_* \to \mathbb{Z}$ is acyclic. Let us consider $\tau C_* = (\operatorname{Ker}\epsilon \to C_0)$ the truncated complex concentrated in degrees 1 and 0. The natural quotient map $C_* \to \tau C_*$ allows to compare two spectral sequences. The first one is associated to the double complex $P_* \otimes_{SL_2\mathbb{C}} \tau C_*$ and converges to the homology of $SL_2\mathbb{C}$ (use the vertical differential first). As it is concentrated on the two bottom lines it yields a long exact sequence, like the Wang sequence. Unscrewing the connecting homomorphism one recognizes the long exact sequence in homology of the cofibration $BBo \to BSL_2\mathbb{C} \to Cof(i_{Bo})$. The second one, associated to the double complex $P_* \otimes_{SL_2\mathbb{C}} C_*$, yields in low degrees the classical Bloch-Wigner sequence (see [DS82]). In particular we get isomorphisms $H_3(Cof(i_{Bo});\mathbb{Z}) \cong \mathcal{P}(\mathbb{C})$ and $\operatorname{Im}(H_3(\mathbb{C}^*;\mathbb{Z}) \to H_3(SL_2\mathbb{C};\mathbb{Z})) \cong \mathbb{Q}/\mathbb{Z}$. $\square$

3. Lifting the Bloch invariant, the compact case

We construct in this section a class in $K_3(\mathbb{C})$ for every closed, compact, oriented hyperbolic 3-manifold and show it coincides with the Neumann-Yang *Bloch invariant*, [NY99].

3.1. Spin structures on hyperbolic manifolds

Usually, a Spin structure on a 3-dimensional oriented metric manifold M is a lift of the classifying map $M \to BSO(3)$ to $BSpin(3)$. Equivalently, since the inclusions of the maximal compact subgroups $SO(3) \subset PSL_2(\mathbb{C})$ and $Spin(3) \subset SL_2(\mathbb{C})$ are homotopy equivalences, one can consider lifts of the composite map $M \to BSO(3) \to BPSL_2(\mathbb{C})$ to $BSL_2(\mathbb{C})$. Now, in the case M is an oriented hyperbolic manifold, M has the homotopy type of the classifying space of the discrete group $\Gamma = \pi_1 M$ and its structural map is a canonical representation of Γ as a discrete and cocompact subgroup of $PSL_2(\mathbb{C})$. Fixing such an inclusion we get a map $M = B\Gamma \to BPSL_2(\mathbb{C})$, which is a representative of the classifying map of the tangent bundle. In this context, a Spin structure on M is a lift of this map to $BSL_2(\mathbb{C})$. Hence, the set of Spin structures which are compatible with the hyperbolic structure is in one-to-one correspondence with the set of group homomorphisms $\Gamma \to SL_2(\mathbb{C})$ that lift the structural morphism $\Gamma \hookrightarrow PSL_2(\mathbb{C})$, [CS83]. Let us fix such a Spin structure $\rho : \Gamma \to SL_2\mathbb{C}$.

3.2. The invariant $\gamma(M)$

We start with an $\mathbb{S}$-orientation $t \in \mathbb{S}_3(B\Gamma)$ coming from a stable parallelization that extends over the 3-skeleton the Spin-structure ρ (recall from Example 1.1 that ρ corresponds to a trivialization of the normal bundle over the 2-skeleton of M). Note that the reduced homology groups are canonical direct factors of the unreduced ones for pointed spaces, so we have a projection $\mathbb{S}_3(M) \twoheadrightarrow \widetilde{\mathbb{S}}_3(M) \cong \pi_3^s(M)$, sending a given orientation $t \in \mathbb{S}_3(M)$ to a *reduced orientation class* $\tilde{t}$ in $\pi_3^s(M)$.

The idea is to use the structural map ρ to obtain an element in $\pi_3^s(BSL_2\mathbb{C})$. Then include $SL_2\mathbb{C}$ into the infinite special linear group $SL\mathbb{C}$. This defines for us an element in

$$\pi_3^s(BSL\mathbb{C}) \cong \pi_3^s(BSL\mathbb{C}^+).$$

Lemma 3.1 *The stabilization map $\pi_3 BSL\mathbb{C}^+ \to \pi_3^s BSL\mathbb{C}^+$ is an isomorphism.*

Proof: Since $BSL\mathbb{C}^+$ is simply connected, Freudenthal's suspension theorem tells us that the stabilization homomorphism $\pi_3 BSL\mathbb{C}^+ \twoheadrightarrow \pi_3^s BSL\mathbb{C}^+$ is an epimorphism. The infinite loop space $BSL\mathbb{C}^+$ is the universal cover of $BGL\mathbb{C}^+$

and its associated spectrum $K\mathbb{C}\langle 1\rangle$ is the 1-connected cover of the K-theory spectrum $K\mathbb{C}$. The map of spectra $\Sigma^\infty BSL\mathbb{C}^+ \to K\mathbb{C}\langle 1\rangle$, adjoint to the identity, yields a right inverse to the stabilization map, which must therefore be a monomorphism. □

Definition 3.2 Let M be a closed, compact, orientable hyperbolic 3-manifold with fundamental group Γ (thus $M \simeq B\Gamma$). Fix a Spin-structure $\rho : \Gamma \to SL_2\mathbb{C}$ and a reduced stable orientation $t \in \pi_3^s(B\Gamma)$ coming from a stable parallelization extending ρ. The element $\gamma(M)$ is then the image of $\tilde{t}$ by the homomorphism

$$\pi_3^s(B\Gamma) \xrightarrow{\rho_*} \pi_3^s(BSL_2\mathbb{C}) \xrightarrow{i} \pi_3^s(BSL\mathbb{C}) \cong \pi_3^s(BSL\mathbb{C}^+) \xrightarrow{\cong} K_3(\mathbb{C}).$$

3.3. Independence from the p_1-structure

The preceding definition apparently depends on the choice of the orientation. We prove here that $\gamma(M)$ is completely determined by the Spin-structure only.

Lemma 3.3 *Let M be a closed orientable manifold of dimension d and $c_{(2)} : Th(\nu_M) \to \Sigma^d\mathbb{S}$ be the map obtained by collapsing the 2-skeleton of M. The S-dual map of $c_{(2)}$ is then, up to sign, the map $i_c : \Sigma^{-N}\mathbb{S} \to \Sigma^{-d}\Sigma^\infty M_+$ induced by the inclusion of the center of the top-dimensional cell.*

Proof: The two duality maps we consider are $u : \mathbb{S} \to Th(\nu_M) \wedge \Sigma^{-d}\Sigma^\infty M_+$ and $v : \mathbb{S} \to \mathbb{S}^d \wedge \mathbb{S}^{-d}$. By Definition 1.3, we have to prove that the two maps $(c_{(2)} \wedge 1_{\Sigma^{-d}\Sigma^\infty M_+}) \circ u$ and $(1_{S^d} \wedge i_c) \circ v$ are homotopic, i.e. coincide in

$$[\mathbb{S}, S^d \wedge \Sigma^{-d}\Sigma^\infty M_+] = [\mathbb{S}, \Sigma^\infty M_+] = \pi_0^s(M_+) \cong \mathbb{Z}.$$

The collapse map $M \to pt$ induces an isomorphism $\pi_0^s(M_+) \to \pi_0^s(S^0)$ so we may post–compose with this collapse map. Let us compute the homotopy class of the map $(1_{S^d} \wedge i_c) \circ v$

$$\begin{tikzcd} \mathbb{S} \arrow[r] & \mathbb{S}^d \wedge \mathbb{S}^{-d} \arrow[r] \arrow[rr, bend right, "Id"'] & \mathbb{S}^d \wedge \Sigma^{-d} M^+ \arrow[r] & \mathbb{S}^d \wedge \Sigma^{-d} S^0. \end{tikzcd}$$

Since the duality map v is an equivalence this is a generator of $\pi_0^s(S^0) = \pi_0^s(M^+)$. To compare it with $(C \wedge 1_{\Sigma^{-d}\Sigma^\infty M^+}) \circ u$, we turn back to the definition of the duality map u. One sees that the above composite is the desuspension of the stable map induced by the following map of spaces, where N stands for a sufficiently large integer:

$$S^{d+N} \xrightarrow{c_{(2)}} Th(\nu_M) \xrightarrow{\Delta'} Th(\nu_M) \wedge M^+ \xrightarrow{Id \wedge C} Th(\nu_M) \wedge S^0 \xrightarrow{c_{(2)}} S^{d+N} \wedge S^0 \simeq S^{d+N}$$

This map is equal to the map induced by the collapse of the complement of the tubular neighbourhood of M restricted to the top-dimensional cell. The tubular neighbourhood restricted to the n-th cell is a trivial disc bundle, therefore the collapse map $S^{d+N} \to D^d \times D^N/\partial(D^d \times D^N) = S^{d+N}$ is of degree ± 1. □

Proposition 3.4 *Let M be a closed, compact, orientable hyperbolic 3-manifold. The reduced orientation class $\tilde{t} \in \pi_3^s(B\Gamma)$ is independent of the p_1-structure. Consequently the element $\gamma(M)$ depends only on the Spin-structure.*

Proof: We have a cofibre sequence $Th(\nu_M|_{M^{(2)}}) \to Th(\nu_M) \xrightarrow{c_{(2)}} \Sigma^3\mathbb{S}$ of spectra of finite type. Therefore, by [Rud98, Lemma II.2.10], we have an S-dual cofibre sequence of spectra of finite type $\Sigma^{-3}\mathbb{S} \to \Sigma^{-3}\Sigma^\infty M_+ \to (Th(\nu_M)|_{M^{(2)}})^*$, where the first map has been identified in Lemma 3.3.

As a consequence we have a commutative diagram, where the vertical arrows are induced by S-duality:

$$\begin{tikzcd}
\mathbb{S}^0(S^3) \arrow[r] \arrow[d] & \mathbb{S}^0(Th\nu_M) \arrow[r] \arrow[d] & \mathbb{S}^0(Th(\nu_m|_{M^{(2)}})) \arrow[d] \\
\pi_3^s(\mathbb{S}) \arrow[r] & \pi_3^s(M_+) \arrow[r] & \mathbb{S}_0(Th(\nu_M|_{M^{(2)}})^*)
\end{tikzcd}$$

The map $\mathbb{S} \to \Sigma^\infty M_+$ splits so that the bottom row is a short exact sequence and we can identify $\mathbb{S}_0(Th(\nu_M|_{M^{(2)}})^*)$ with $\pi_3^s(M)$. The diagram shows that the reduced orientation class $\tilde{t} \in \pi_3^s(M)$ is S-dual to the cohomological orientation class restricted to the 2-skeleton, which is unaffected by a change of p_1-structure. □

Remark 3.5 At this point the class $\gamma(M)$ could be independent of the Spin structure, even though its construction is not. We will return to this question in Proposition 3.7 hereafter.

3.4. Comparison with the Bloch invariant

Let us recall how Neumann and Yang construct in [NY99] the Bloch invariant $\beta(M) \in \mathcal{B}(\mathbb{C})$. The later is the kernel of the morphism $\mathcal{P}(\mathbb{C}) \to \Lambda^2(\mathbb{C}^*/\mu_\mathbb{C})$ in the Bloch-Wigner exact sequence, Proposition 2.6. Since M is oriented hyperbolic, $\Gamma \subset PSL_2\mathbb{C}$ and M can be identified with the quotient $\mathcal{H}^3/\Gamma$. There is a preferred cohomology class in $H^3(PSL_2\mathbb{C};\mathcal{P}(\mathbb{C}))$ constructed as follows. Pick a point $* \in \partial\mathcal{H}$ and to each symbol $[A|B|C]$ with $A, B, C \in PSL_2\mathbb{C}$ associate the class of the ideal tetrahedron $(*, A*, AB*, ABC*)$ (if this happens to be flat, its class is 0). The defining relation in $\mathcal{P}(\mathbb{C})$ implies exactly that this is a 3-cocycle on the group $PSL_2\mathbb{C}$. An easy computation shows that its cohomology class is independent of the choice of $*$ in $\partial\mathcal{H}$ (see for instance [Dup01, Chapter 8]). By the universal coefficient theorem this gives a map $H_3(PSL_2\mathbb{C};\mathbb{Z}) \to \mathcal{P}(C)$.

By [NY99, Proposition 4.3] the invariant $\beta(M)$ coincides with the image of the fundamental class through the composite

$$H_3(M;\mathbb{Z}) \longrightarrow H_3(PSL_2\mathbb{C};\mathbb{Z}) \longrightarrow \mathcal{P}(\mathbb{C}).$$

This proves that $\beta(M)$ is well-defined, and lies indeed in $\mathcal{B}(\mathbb{C})$.

Theorem 3.6 *Let M be a closed, compact, orientable hyperbolic 3-manifold. The element $\gamma(M)$ lifts the Bloch invariant $\beta(M)$.*

Proof: It is well-known that the cokernel of the natural map from Milnor's K-theory $K_3^M(\mathbb{C}) \to K_3(\mathbb{C})$ provides a splitting for $H_3(SL_2\mathbb{C};\mathbb{Z}) \to K_3(\mathbb{C})$. Moreover the morphism $H_3(SL_2\mathbb{C};\mathbb{Z}) \to \mathcal{P}(\mathbb{C})$ factors through $H_3(PSL_2\mathbb{C};\mathbb{Z})$. According to the construction of the Bloch-Wigner map $\mathrm{bw}\colon K_3(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ (see [Sus90, Section 5]) we have a commutative diagram

$$\begin{tikzcd}
\pi_3^s(M) \arrow[r, "\rho"] \arrow[d] & \pi_3^s(BSL_2\mathbb{C}) \arrow[r] \arrow[d] & K_3(\mathbb{C}) \arrow[d] \\
H_3(M;\mathbb{Z}) \arrow[r] & H_3(PSL_2\mathbb{C};\mathbb{Z}) \arrow[r] & \mathcal{P}(\mathbb{C})
\end{tikzcd}$$

and obviously the reduced $\mathbb{S}$-orientation $\tilde{\iota}$ maps to an orientation in $H_3(M;\mathbb{Z})$. □

As a consequence of Proposition 3.4 the class $\gamma(M) \in K_3(\mathbb{C})$ depends at most on the Spin structure chosen on M. A change of Spin structure is parametrized by a $\mathbb{Z}/2$ cohomology class and, as the following proposition shows, this results at most in the addition to $\gamma(M)$ of the unique non trivial $\mathbb{Z}/2$ class in $K_3(\mathbb{C})$.

Proposition 3.7 *Choose two different Spin structures $\rho, \rho' : \pi_1(M) \to SL_2(\mathbb{C})$ that extend the same homological orientation of M. Apply the preceding constructions to two parallelizations of M extending these Spin structures and call $\gamma(M)$ and $\gamma'(M)$ the resulting K-theory classes. Then $\gamma(M) - \gamma'(M) \in \mathbb{Z}/2 \subset K_3(\mathbb{C})$ and by suitably changing the Spin structure one can make the difference non–zero.*

Proof: Let $\tilde{\iota} \in \pi_3^s(M)$ denote any reduced orientation class of M. Since by Lemma 2.5 $\pi_3^s(SL_2\mathbb{C}) \simeq H_3(SL_2\mathbb{C};\mathbb{Z})$ the following commutative diagram

$$\begin{tikzcd}
\pi_3^s(M) \arrow[r] \arrow[d] & \pi_3^s(SL_2\mathbb{C}) \arrow[d, "\wr"] \\
H_3(M) \arrow[r] & H_3(SL_2\mathbb{C};\mathbb{Z})
\end{tikzcd}$$

shows that the class $\gamma(M) = \rho_*(\tilde{t})$ depends on Spin structure only via the morphism ρ. Since ρ and ρ' extend the same homological orientation there exists a non-trivial homomorphism $\sigma : \pi_1(M) \to \mathbb{Z}/2$ such that $\rho = \rho' + \sigma$ (notice that this notation makes sense since $\mathbb{Z}/2 \subset SL_2\mathbb{C}$ is central). Denote by $t \in H_3(M;\mathbb{Z})$ the homological orientation that both ρ and ρ' extend. Then $(\rho_* - \rho'_*)(t) = \sigma_*(t) \in H_3(SL_2\mathbb{C};\mathbb{Z})$ factors via $H_3(\mathbb{Z}/2;\mathbb{Z}) = \mathbb{Z}/2 \hookrightarrow H_3(SL_2\mathbb{C};\mathbb{Z})$. To show that this difference may not be zero it is enough to prove that one can choose the Spin structure, i.e. the morphism $\pi_1(M) \to \mathbb{Z}/2$ in such a way that the image of the fundamental class of M is non-zero in $H_3(\mathbb{Z}/2;\mathbb{Z})$. An explicit construction may be given as follows: the 3-skeleton of $B\mathbb{Z}/2$ is the projective space $\mathbb{R}P^3$, and the non-zero class of $H_3(B\mathbb{Z}/2\mathbb{Z};\mathbb{Z})$ is represented by the orientation class of the projective space. Since M is (homologically) oriented the collapse of the 2-skeleton map $M \to M/M^{(2)} \simeq S^3$ is an orientation-preserving map. The composition with the canonical maps $S^3 \to RP^3 \hookrightarrow B\mathbb{Z}/2$ is a map $K(\pi_1(M),1) = M \to B\mathbb{Z}/2$ which by construction sends the fundamental class of M onto the non-zero class in $H_3(B\mathbb{Z}/2;\mathbb{Z})$. □

Remark 3.8 Our approach can be applied in higher dimensions, since the same definition can be used in a straightforward manner to define a class in $K_n(\mathbb{C})$ associated to an n-dimensional $\mathbb{S}$-oriented hyperbolic manifold. This definition might of course depend on the chosen orientation in general, if it exists.

Borel defined in [Bor77] the Borel regulator bo-reg : $K_3(\mathbb{C}) \to \mathbb{R}$. Likewise the Bloch regulator is a map bl-reg : $\mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$ and the image of the Bloch-Wigner map bw : $K_3(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ lies in $\mathcal{B}(\mathbb{C})$ (see [Sus90]).

Corollary 3.9 *Let M be a closed compact oriented hyperbolic manifold of dimension* 3 *with fundamental group* Γ. *Then, to a Spin-structure ρ corresponds, in a canonical way, a class $\gamma(M) \in K_3(\mathbb{C})$ such that the hyperbolic volume of M is determined by the equality*

$$\text{bo-reg}(\gamma(M)) = \frac{\text{vol}(M)}{2\pi^2}.$$

Furthermore the Chern-Simons invariant $\text{CS}(M)$ *is determined by the congruence*

$$\mu(\gamma(M)) \equiv \frac{-CS(M) + i \cdot \text{vol}(M)}{2\pi^2} \pmod{\mathbb{Q}}.$$

Proof: This follows directly from Theorem 3.6. Neumann and Yang prove in [NY99, Theorem 1.3] that one can recover the volume and the Chern-Simons invariant via the Borel and Bloch regulators. □

4. Lifting the Bloch invariant, the non-compact case

Let M be a non-compact, orientable, hyperbolic 3-manifold of finite volume with $\Gamma = \pi_1(M)$. Since M has finite volume it has a finite number of cusps and all of them are toroidal, [Rat94, Theorem 10.2.1]. Choose such a cusp $x \in M$ and denote by $Bo \subset SL_2\mathbb{C}$ the preimage of the stabilizer of x. As in Subsection 2.2, i_{Bo} denotes the map $BBo \to BSL_2\mathbb{C}$. Notice that since the stabilizers of all cusp points are conjugate in $SL_2\mathbb{C}$, the homotopy type of the cofibre $Cof(i_{Bo})$, is independent of the choice of x. Choose a Spin-structure on M, i.e. a homomorphism $\rho : \Gamma \to SL_2\mathbb{C}$ lifting the canonical representation in $PSL_2(\mathbb{C})$. The representation ρ contains parabolic elements, i.e. elements fixing a point in the boundary $\partial\overline{\mathcal{H}}^3$. Choose a sufficiently small δ-horosphere around each cusp of M and denote by M_δ the compact submanifold obtained by removing these horospheres from the cusps of M, [Thu97, Theorem 4.5.7]. The inclusion $M_\delta \hookrightarrow M$ is a homotopy equivalence.

4.1. *A first indeterminacy for $\gamma(M)$*

Let $T \subset \partial M_\delta$ denote any component of the boundary, so $T \simeq S^1 \times S^1$. Consider the composite

$$T = B(\mathbb{Z}^2) \hookrightarrow \partial M_\delta \hookrightarrow M_\delta \xrightarrow{B\rho} BSL_2\mathbb{C} \longrightarrow Cof(i_{Bo}).$$

As the action of $SL_2\mathbb{C}$ is transitive on the boundary of the hyperbolic space, all stabilizers of points in $\partial\mathcal{H}^3$ are conjugate. The inclusion of $\mathbb{Z}\oplus\mathbb{Z}$ into $SL_2\mathbb{C}$ is then conjugate to an inclusion into Bo, so that the map $T \to Cof(i_{Bo})$ is null-homotopic. The difference between two null-homotopies induced by such conjugations is given by conjugating by an element of the Borel group Bo and this latter map is homotopic to the identity, as a self-map of the pair $(BSL_2\mathbb{C}, BBo)$.

So, from the choice of the Spin-structure, we get a map $M_\delta/\partial M_\delta \to Cof(i_{Bo})$, which is well-defined up to homotopy. A stable parallelization of the tangent bundle of M_δ gives rise to a fundamental class $t \in \mathbb{S}_3(M_\delta, \partial M_\delta) \cong \pi_3^s(M_\delta/\partial M_\delta)$. Pushing this class by the above map, we get a well-defined class $\gamma_P(M) \in \pi_3^s(Cof(i_{Bo}))$.

Theorem 4.1 *Let M be a non-compact, orientable, hyperbolic 3-manifold of finite volume. It is then always possible to lift the class $\gamma_P(M)$ to a class $\gamma(M) \in K_3(\mathbb{C})$, and there are $\mathbb{Q}/\mathbb{Z}$ possible lifts.*

Proof: According to Proposition 2.6, the class $\gamma_P(M)$ lives in $\mathcal{P}(\mathbb{C}) \cong \pi_3^s(Cof(i_{Bo}))$. Thus our invariant $\gamma_P(M)$ coincides in fact with the Bloch invariant $\beta(M)$, defined in an analogous way to the compact case. We wish to lift it through the connecting homomorphism $\delta : \pi_3^s(BSL_2\mathbb{C}) \to \pi_3^s(Cof(i_{Bo}))$.

According to Neumann and Yang, [NY99, Section 5] the Bloch invariant is the scissors congruence class of any hyperbolic ideal triangulation of M and this class belongs to the kernel $\mathcal{B}(\mathbb{C})$ of $\mathcal{P}(\mathbb{C}) \to \Lambda^2(\mathbb{C}^*/\mu_{\mathbb{C}})$.

The existence of the lift follows at once from Proposition 2.6. This explains the $\mathbb{Q}/\mathbb{Z}$ indeterminacy: the image of the map $\pi_3^s(BBo) \to \pi_3^s(BSL_2\mathbb{C})$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Now it suffices to push any lift to $\pi_3^s(BSL\mathbb{C})$, a group isomorphic to $K_3(\mathbb{C})$ by Lemma 3.1. □

Remark 4.2 The fact that the Bloch invariant lies in $\mathcal{B}(\mathbb{C})$ has a nice geometrical interpretation. Hyperbolic tetrahedra up to isometry are in one to one correspondence with elements of $\mathbb{C} - \{0,1\}$, the modulus of the tetrahedron. If one starts with a collection of such tetrahedra and wants to glue them to a hyperbolic space then a theorem of Thurston says that the moduli of the tetrahedra have to satisfy a compatibility relation in $\Lambda^2(\mathbb{C} - \{0,1\})$, namely $\Sigma(z \wedge (1-z)) = 0$. The above morphism $\mathcal{P}(\mathbb{C}) \to \Lambda^2(\mathbb{C}^*/\mu_{\mathbb{C}})$ is $z \mapsto 2(z \wedge (1-z))$. In particular the image under this morphism of an ideal triangulation of the hyperbolic manifold M will be trivially 0 since we started with a hyperbolic manifold.

Theorem 4.1 immediately provides the following.

Corollary 4.3 *[Gon99, Theorem 1.1] Let M be a non-compact, orientable, hyperbolic 3-manifold of finite volume. Then M defines naturally a class $\gamma(M) \in K_3(\mathbb{C}) \otimes \mathbb{Q}$ such that* $\text{bo-reg}(\gamma(M)) = \frac{\text{vol}(M)}{2\pi^2}$. □

A. Orientation with respect to algebraic K-theory

To generalize this approach to higher dimensional manifolds, one cannot follow the same strategy, as it is not known whether or not all hyperbolic manifolds are stably parallelizable. There is however an intermediate homology theory, between stable homotopy and ordinary homology. What we have done in the three dimensional situation was to start with an $\mathbb{S}$-orientation, whereas the former approaches [Gon99], [NY99], and [CMJ03] all roughly started from the fundamental class in homology.

The first author's original insight to the question of lifting the Bloch invariant was to work with $K\mathbb{Z}$-orientation, where $K\mathbb{Z}$ denotes the connective spectrum of the algebraic K-theory of the integers. We believe that this is a point of view which is close enough to ordinary homology (or topological K-theory) so as to be able to do computations, but at the same time not too far away from the stable homotopy so that the above techniques to construct an invariant in $K_3(\mathbb{C})$ can go through.

In his foundational paper [Lod76] Loday defines a product in algebraic K-theory by means of a pairing of spectra (in the sense of Whitehead). Given two rings R and S, consider the connective Ω-spectra KR and KS corresponding to the infinite loop spaces $BGLR^+ \times K_0R$ and $BGLS^+ \times K_0S$ respectively (the deloopings are given by the spaces $BGL(S^nR)^+$ where SR denotes the suspension of the ring R). Then there exists a pairing

$$\star : KS \wedge KR \to K(S \otimes R).$$

We will be interested in the case when $S = \mathbb{Z}$. In this case the pairing goes to KR. The pairing includes in particular compatible maps

$$BGL(S^n\mathbb{Z})^+ \wedge BGLR^+ \to BGL(S^n\mathbb{Z} \otimes R)^+ = BGL(S^nR)^+$$

which yield a map of spectra $\star : K\mathbb{Z} \wedge BGLR^+ \to KR$. In order to compare the present construction with the previous one based on an $\mathbb{S}$-orientation, we will need to understand the map obtained by precomposing with $\varepsilon \wedge 1$, where $\varepsilon : \mathbb{S} \to K\mathbb{Z}$ is the unit of the ring spectrum $K\mathbb{Z}$. We first look at the global pairing of spectra.

Lemma A.1 *The composite map* $\mathbb{S} \wedge KR \xrightarrow{\varepsilon\wedge 1} K\mathbb{Z} \wedge KR \xrightarrow{\star} KR$ *is the identity.*

Proof: We learn from May, [May80], that KR is a ring spectrum. In particular the composite $\mathbb{S} \wedge KR \xrightarrow{\varepsilon\wedge 1} KR \wedge KR \xrightarrow{\star} K(R \otimes R) \xrightarrow{\mu} KR$ is the identity. By naturality and using the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ we see that the map in question must be the identity as well. □

We are interested in the infinite loop space $BGLR^+$ and wish to compare it to the spectrum KR. For that purpose we use the pair of adjoint functors

$$\Sigma^\infty : Spaces \leftrightarrows Spectra : \Omega^\infty,$$

where $\Sigma^\infty X = \mathbb{S} \wedge X$ is the suspension spectrum of the space X and $\Omega^\infty E$ is the 0th term of the Ω-spectrum representing the cohomology theory E^*. If E is an Ω-spectrum, then $\Omega^\infty E = E_0$ and we write $a : \mathbb{S} \wedge E_0 \to E$ for the adjoint of the identity.

Proposition A.2 *The composite map* $\mathbb{S} \wedge BGLR^+ \xrightarrow{\varepsilon\wedge 1} K\mathbb{Z} \wedge BGLR^+ \xrightarrow{\star} KR$ *is homotopic to* $a : \mathbb{S} \wedge BGLR^+ \to KR$.

Proof: We consider the commutative diagram

$$\begin{tikzcd}
\mathbb{S} \wedge \mathbb{S} \wedge BGLR^+ \arrow[r, "\varepsilon\wedge 1\wedge 1"] \arrow[d, "1\wedge a"'] & K\mathbb{Z} \wedge \mathbb{S} \wedge BGLR^+ \arrow[d, "1\wedge a"] \arrow[dr, "\star"] & \\
\mathbb{S} \wedge KR \arrow[r, "\varepsilon\wedge 1"'] & K\mathbb{Z} \wedge KR \arrow[r, "\star"'] & KR.
\end{tikzcd}$$

The square is obviously commutative and the triangle commutes up to homotopy since the Loday product $\star$ forms a Whitehead pairing, [Lod76, p.346]. □

Thus we can recover the invariant $\gamma(M)$ as follows. Consider the composite

$$h : K\mathbb{Z} \wedge M \xrightarrow{1\wedge B\rho} K\mathbb{Z} \wedge BSL_2\mathbb{C} \longrightarrow K\mathbb{Z} \wedge BGL\mathbb{C}^+ \xrightarrow{\star} K\mathbb{C}.$$

Proposition A.3 *Let M be a closed, compact, orientable hyperbolic 3-manifold and choose a $K\mathbb{Z}$-orientation $s \in K\mathbb{Z}_3(M) \cong \pi_3(K\mathbb{Z} \wedge M)$. The invariant $\gamma(M) \in K_3(\mathbb{C})$ is then equal to $h_*(s)$.* □

Between the $K\mathbb{Z}$-orientation and the invariant $\gamma(M)$ there is an interesting class in $K_3(\mathbb{Z}\Gamma)$. It is obtained as the image of the $K\mathbb{Z}$-orientation under the composite

$$K\mathbb{Z}_3(B\Gamma) \longrightarrow K\mathbb{Z}_3(BGL(\mathbb{Z}\Gamma)^+) \longrightarrow K_3(\mathbb{Z}\Gamma),$$

where the first arrow is induced by the canonical inclusion $\Gamma \hookrightarrow GL_1(\mathbb{Z}\Gamma)$ and the second is a Loday product. It is not difficult to see that we recover $\gamma(M)$ by further composing with

$$K_3(\mathbb{Z}\Gamma) \xrightarrow{\rho_*} K_3(\mathbb{Z}SL_2\mathbb{C}) \longrightarrow K_3(M_2\mathbb{C}) \cong K_3(\mathbb{C}).$$

The second arrow is the fusion map, which takes the formal sum of invertible matrices to the actual sum in $M_2\mathbb{C}$. The final isomorphism is just Morita invariance. In particular if we turn back to the general questions stated in the introduction, we may say that the scissor congruence class of a compact hyperbolic three manifold $M = B\Gamma$ is encoded as an orientation class in $K\mathbb{Z}_3(B\Gamma)$. Moreover this might lead to explicit computations of volumes, and maybe Chern-Simons invariants, for, as the Borel regulator is "known" and the above maps are explicit, it remains to find an explicit representative for the "orientation class" in $K\mathbb{Z}_3(B\Gamma)$, which could be done along the lines of [MO02].

REFERENCES

ABS64. M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology* **3** (suppl. 1):3–38, 1964.

Ada66. J. F. Adams. On the groups $J(X)$. IV. *Topology* **5**:21–71, 1966.

Ada74. J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

Bor77. A. Borel. Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4**(4):613–636, 1977.

CMJ03. J. L. Cisneros-Molina and J. D. S. Jones. The Bloch invariant as a characteristic class in $B(\mathrm{SL}_2(\mathbb{C}),\mathcal{T})$. *Homology Homotopy Appl.* **5**(1):325–344 (electronic), 2003.

CS83. M. Culler and P. B. Shalen. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math. (2)* **117**(1):109–146, 1983.

DS82. J. L. Dupont and C. H. Sah. Scissors congruences. II. *J. Pure Appl. Algebra* **25**(2):159–195, 1982.

Dup01. J. L. Dupont. *Scissors congruences, group homology and characteristic classes*, volume 1 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

DZ06. J. L. Dupont and C. K. Zickert. A dilogarithmic formula for the Cheeger-Chern-Simons class. *Geom. Topol.* **10**:1347–1372 (electronic), 2006.

FJ93. F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic K-theory. *J. Amer. Math. Soc.* **6**(2):249–297, 1993.

GZ07. S. Goette and C. K. Zickert. The extended Bloch group and the Cheeger-Chern-Simons class. *Geom. Topol.* **11**:1623–1635 (electronic), 2007.

Gon99. A. Goncharov. Volumes of hyperbolic manifolds and mixed Tate motives. *J. Amer. Math. Soc.* **12**(2):569–618, 1999.

Lod76. J.-L. Loday. K-théorie algébrique et représentations de groupes. *Ann. Sci. École Norm. Sup. (4)* **9**(3):309–377, 1976.

May80. J. P. May. Pairings of categories and spectra. *J. Pure Appl. Algebra* **19**:299–346, 1980.

MO02. M. Matthey and H. Oyono-Oyono. Algebraic K-theory in low degree and the Novikov assembly map. *Proc. London Math. Soc. (3)* **85**(1):43–61, 2002.

MS60. J. Milnor and E. Spanier. Two remarks on fiber homotopy type. *Pacific J. Math.* **10**:585–590, 1960.

MS74. J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies **76**.

Neu04. W. D. Neumann. Extended Bloch group and the Cheeger-Chern-Simons class. *Geom. Topol.* **8**:413–474 (electronic), 2004.

NY99. W. D. Neumann and J. Yang. Bloch invariants of hyperbolic 3-manifolds. *Duke Math. J.* **96**(1):29–59, 1999.

Rat94. J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, *Graduate Texts in Mathematics* **149**. Springer-Verlag, New York, 1994.

Rud98. Y. B. Rudyak. *On Thom spectra, orientability, and cobordism.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller.

Sus84. A. A. Suslin. *Homology of* GL_n*, characteristic classes and Milnor* K*-theory* in Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), *Lecture Notes in Mathematics* **1046**. Springer-Verlag, Berlin, 1984.

Sus90. A. A. Suslin. K_3 of a field, and the Bloch group. *Trudy Mat. Inst. Steklov.* **183**:180–199, 229, 1990. Translated in Proc. Steklov Inst. Math. 1991, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications (Russian).

Swi02. R. M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original.

Thu97. W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, *Princeton Mathematical Series* **35** . Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

MICHEL MATTHEY

deceased

WOLFGANG PITSCH pitsch@mat.uab.es

Departament de Matemàtiques
Universitat Autònoma de Barcelona
E–08193 Bellaterra
Spain

JÉRÔME SCHERER jscherer@mat.uab.es

Departament de Matemàtiques
Universitat Autònoma de Barcelona
E–08193 Bellaterra
Spain

Received: July 17, 2008