

TOPOLOGICAL COMPLEXITY OF H -SPACES

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ABSTRACT. Let X be a (not-necessarily homotopy-associative) H -space. We show that $\mathrm{TC}_{n+1}(X) = \mathrm{cat}(X^n)$, for $n \geq 1$, where $\mathrm{TC}_{n+1}(-)$ denotes the so-called higher topological complexity introduced by Rudyak, and $\mathrm{cat}(-)$ denotes the Lusternik-Schnirelmann category. We also generalize this equality to an inequality, which gives an upper bound for $\mathrm{TC}_{n+1}(X)$, in the setting of a space Y acting on X .

1. INTRODUCTION

The notion of topological complexity, introduced by Farber in [4], is motivated by the study of motion planning algorithms in the field of topological robotics. See also [5, 6] for good introductions to the topic and its context. In this paper we prove results that allow for effective computation of topological complexity, and of higher analogues of the same, in important cases.

We begin with some notation and basic definitions. We use $\mathrm{cat}(X)$ to denote the Lusternik-Schnirelmann (L-S) category of X (normalised, so that $\mathrm{cat}(S^n) = 1$). That is, $\mathrm{cat}(X)$ is the smallest n for which there is an open covering $\{U_0, \dots, U_n\}$ by $n+1$ open sets, each of which is contractible in X . See [2] for a general introduction to L-S category and related topics. The *sectional category* of a fibration $p: E \rightarrow B$, denoted by $\mathrm{secat}(p)$, is the smallest number n for which there is an open covering $\{U_0, \dots, U_n\}$ of B by $n+1$ open sets, for each of which there is a local section $s_i: U_i \rightarrow E$ of p , so that $p \circ s_i = j_i: U_i \rightarrow B$, where j_i denotes the inclusion. This notion generalizes that of L-S category: if E is contractible, then we have $\mathrm{secat}(p) = \mathrm{cat}(B)$; in general we have $\mathrm{secat}(p) \leq \mathrm{cat}(B)$ (see [2, Prop. 9.14]). Now let PX denote the space of (free) paths on a space X ; thus we have $PX = \mathrm{Map}(I, X)$. There is a fibration $P_2: PX \rightarrow X \times X$, which evaluates a path at the initial and final points: for $\alpha \in PX$, we have $P_2(\alpha) = (\alpha(0), \alpha(1))$. This is a fibrational substitute for the diagonal map $\Delta: X \rightarrow X \times X$. We define the *topological complexity* $\mathrm{TC}(X)$ of X to be the sectional category $\mathrm{secat}(P_2)$ of this fibration. That is, $\mathrm{TC}(X)$ is the smallest number n for which there is an open cover $\{U_0, \dots, U_n\}$ of $X \times X$ by $n+1$ open sets, for each of which there is a local section

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$s_i: U_i \rightarrow PX$ of P_2 , i.e., for which $P_2 \circ s_i = j_i: U_i \rightarrow X \times X$, where j_i denotes the inclusion.

We indicate briefly the motivation for topological complexity mentioned above; for a full discussion, see [4, 5, 6]. A point in $X \times X$ is a pair (x, y) of points in X . We imagine X as the space of all states of some system, and x and y as initial and final states of the system. Then a section $s: X \times X \rightarrow PX$ of $P_2: PX \rightarrow X \times X$ may be viewed as a continuous, deterministic association of a path in X , from initial state x to final state y , to each pair of states (x, y) : a *continuous motion planner*. As is easily shown, however, a global section is possible exactly when X is contractible. Since one is naturally interested in state spaces that have interesting topology, i.e., which are noncontractible, one is necessarily forced into considering *discontinuous motion planners*, which correspond to a covering and local sections exactly as in our definition for $\text{TC}(X)$.

We also consider the “higher analogues” of topological complexity introduced by Rudyak in [13] (see also [14] and [1]). For this notion, let $n \geq 2$ and consider the fibration

$$P_n: PX \rightarrow X \times \cdots \times X = X^n,$$

defined by dividing the unit interval $I = [0, 1]$ into $n - 1$ subintervals of equal length, with n subdivision points $t_0 = 0, t_1 = 1/(n - 1), \dots, t_{n-1} = 1$ (thus $n - 2$ subdivision points interior to the interval), and then evaluating at each of the n subdivision points; thus:

$$P_n(\alpha) = (\alpha(0), \alpha(t_1), \dots, \alpha(t_{n-2}), \alpha(1)),$$

for $\alpha \in PX$. This is a fibrational substitute for the n -fold diagonal $\Delta_n: X \rightarrow X^n$. Then the *higher topological complexity* $\text{TC}_n(X)$ is defined as $\text{TC}_n(X) = \text{secat}(P_n)$.

The ordinary topological complexity, then, corresponds to the case in which $n = 2$: $\text{TC}_2(X) = \text{TC}(X)$. Higher topological complexity may readily be motivated in terms of motion planners. A point in X^n is an n -tuple $(x, x_2, \dots, x_{n-1}, y)$. A section (either local or global) of P_n corresponds to a deterministic association of a path in the state space X , from initial state x to final state y , and passing through the specified intermediate states x_2, \dots, x_{n-1} in that order.

Our first main result is as follows (see Section 3):

Theorem 1. *Let X be a connected, CW H -space. Then, for $n \geq 1$, we have $\text{TC}_{n+1}(X) = \text{cat}(X^n)$. In particular, we have $\text{TC}(X) = \text{cat}(X)$.*

Since the value of $\text{cat}(X^n)$ is known, or can be determined, in certain cases in which X is a Lie group or a more general H -space, this result allows the (higher) topological complexity to be determined exactly in those cases (see Examples 3.3 for a discussion). This is of great value, since $\text{TC}(X)$, like $\text{cat}(X)$, proves to be a very delicate invariant to compute and its exact value is known in comparatively few cases.

In the case in which G is a topological group, the equality $\text{TC}(G) = \text{cat}(G)$ appears as [5, Lem. 8.2] and its generalization $\text{TC}_{n+1}(G) = \text{cat}(G^n)$ appears as [1, Th. 3.5]. The proofs of both results make essential use of associativity and existence of (two-sided) inverses. Theorem 1 extends these results to the more general case in which X is an H -space, with no assumption on associativity.

Our second main result is the following (see Section 4):

Theorem 2. *Let $A: X \times Y \rightarrow X$ be an action of Y on X with strict unit. For $n \geq 1$, let $\text{sh}_{n+1}: X \times Y^n \rightarrow X^{n+1}$ denote the shear map defined by*

$$\text{sh}_{n+1} = (p^1, A \circ (p^1, p^2), \dots, A \circ (p^1, p^{n+1})).$$

Here, p^j denotes projection on the j th factor: $p^1: X \times Y^n \rightarrow X$ projects onto X , but for $j = 2, \dots, n + 1$, $p^j: X \times Y^n \rightarrow Y$ projects onto Y . Then we have

$$\text{TC}_{n+1}(X) \leq (\text{secat}(\text{sh}_{n+1}) + 1)(\text{cat}(Y^n) + 1) - 1.$$

In particular, if $n = 1$, then we have $\text{TC}(X) \leq (\text{secat}(\text{sh}) + 1)(\text{cat}(Y) + 1) - 1$, where $\text{sh}: X \times Y \rightarrow X \times X$ is defined as $\text{sh} = (p^1, A)$.

This gives a general upper bound for the (higher) topological complexity that may be applied in many situations of interest and that, in principle, may improve on other general upper bounds in those situations to which it applies. Theorem 2 is a direct generalization of Theorem 1: if the action $A: X \times Y \rightarrow X$ is given by an H -space X acting on itself by multiplication, then Theorem 2 reduces to Theorem 1.

We finish this introduction with a brief outline of the paper. Section 2 is a short section in which we review some standard facts concerning sectional category used in the sequel. In Section 3 we give basic inequalities concerning $\text{TC}_{n+1}(X)$, and the proof of Theorem 1. We give several examples to illustrate the applicability of Theorem 1. In Section 4 we develop the notion of a shear map coming from an action of one space on another and give the proof of Theorem 2. The final section, Section 5, is a short, informal section in which we give several interesting questions that stem from our results and examples.

2. NOTATIONAL CONVENTIONS AND BASIC FACTS

Our conventions are that cat , respectively secat (hence TC), is *one less than* the number of suitable open sets in a covering of X , respectively, of B . Thus, a space X is contractible exactly when $\text{cat}(X) = 0$, and we have $\text{cat}(S^n) = 1$ for any $n \geq 1$. Furthermore, a fibration $p: E \rightarrow B$ admits a section exactly when $\text{secat}(p) = 0$ (and if E is also contractible, we then recover the case $\text{cat}(B) = 0$). These conventions agree with those used in [1, 2, 7], for example. They entail that $\text{TC}(S^n) = 1$ for n odd and $\text{TC}(S^n) = 2$ for n even. However, some authors use the “un-normalized” version of secat (hence TC) whether or not the “normalized” cat is used. Hence, our $\text{TC}(X)$ will be one less than that of some authors, such as Farber in [5].

There are two well-known and basic facts which we use throughout the article to compare sectional categories of various maps. We state them here; each is easily justified directly from the definitions. First, suppose given a fibration $p: E \rightarrow B$ and any map $f: B' \rightarrow B$. Form the pullback

$$\begin{array}{ccc} Q & \longrightarrow & E \\ q \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B. \end{array}$$

Then we have $\text{secat}(q) \leq \text{secat}(p)$. Second, suppose we are given a fibration $p': E' \rightarrow B'$ and any maps $\theta: Q \rightarrow E'$ and $q: Q \rightarrow B'$ for which the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\theta} & E' \\ & \searrow q & \swarrow p' \\ & & B' \end{array}$$

is homotopy commutative. Then we have $\text{secat}(p') \leq \text{secat}(q)$. In this latter situation, a local section $s: B' \rightarrow Q$ of $q: Q \rightarrow B'$ leads initially to a local homotopy section $\theta \circ s: B' \rightarrow E'$ of $p': E' \rightarrow B'$, which may then be adjusted into a section using the assumption that $p': E' \rightarrow B'$ is a fibration.

A standard application of the pullback square situation is as follows. Suppose that X is a based space with basepoint x_0 . Write $\mathcal{P}X$ for the based path space of maps $\alpha: I \rightarrow X$ with $\alpha(0) = x_0$, let $\mathcal{P}_1: \mathcal{P}X \rightarrow X$ denote the evaluation map $\mathcal{P}_1(\alpha) = \alpha(1)$, and $i_2: X \rightarrow X \times X$ be the inclusion on the second factor, $i_2(x) = (x_0, x)$. The pullback square

$$\begin{array}{ccc} \mathcal{P}X & \longrightarrow & PX \\ \mathcal{P}_1 \downarrow & & \downarrow P_2 \\ X & \xrightarrow{i_2} & X \times X \end{array}$$

yields the inequality $\text{cat}(X) \leq \text{TC}(X)$, since $\text{cat}(X) = \text{secat}(\mathcal{P}_1)$ ($\mathcal{P}X$ is contractible). This can easily be generalized to an inequality concerning the higher topological complexity. Write $I_2: X^n \rightarrow X^{n+1}$ for the inclusion $I_2(x_1, \dots, x_n) = (x_0, x_1, \dots, x_n)$, with $x_0 \in X$ the basepoint, and $\mathcal{P}_n: \mathcal{P}X \rightarrow X^n$ for the map with coordinates given by evaluation at $1/n, 2/n, \dots, 1$. Although (for $n \geq 2$) this is not the usual based path fibration, nonetheless the space $\mathcal{P}X$ is still contractible so that $\text{secat}(\mathcal{P}_n) = \text{cat}(X^n)$.

Lemma 2.1. *Let X be any connected pointed space. Then $\text{cat}(X^n) \leq \text{TC}_{n+1}(X)$ for any $n \geq 1$.*

Proof. Note that we have a pullback square

$$\begin{array}{ccc} \mathcal{P}X & \longrightarrow & PX \\ \mathcal{P}_n \downarrow & & \downarrow P_{n+1} \\ X^n & \xrightarrow{I_2} & X^{n+1}, \end{array}$$

which gives $\text{cat}(X^n) \leq \text{secat}(P_{n+1})$ from the pullback property of secat recalled above. This latter is $\text{TC}_{n+1}(X)$ by definition, and the inequality follows. \square

Finally, we will sometimes refer to the sectional category of an arbitrary map $f: X \rightarrow Y$ (not necessarily a fibration). What we mean by that is that we first replace f by a fibration. The sectional category does not depend on the choice of such a fibrational substitute. This allows us to deal with homotopy pullback squares instead of pulling back a fibration. In particular we may write equalities such as $\text{cat}(X) = \text{secat}(* \rightarrow X)$.

3. (HIGHER) TOPOLOGICAL COMPLEXITY AND H -SPACES

First we observe that, for $n \geq 1$, we have the following basic inequalities, which extend in a natural way those for $\text{TC}(X)$ recalled in Section 1.

Proposition 3.1. *Suppose X is a connected, noncontractible CW complex. For $n \geq 1$, we have*

$$n \leq \text{cat}(X^n) \leq \text{TC}_{n+1}(X) \leq \text{cat}(X^{n+1}) \leq (n + 1)\text{cat}(X).$$

Proof. The first and last of these are standard inequalities concerning $\text{cat}(X)$ (see [2, Th. 1.37, Th. 1.47]; the last, which derives from the product inequality, needs some separation hypothesis on X , which we have covered here by assuming X is CW). The second inequality is Lemma 2.1, and the third follows from the definition of $\text{TC}_{n+1}(X)$ together with the general inequality $\text{secat}(p) \leq \text{cat}(B)$ for any fibration $p: E \rightarrow B$. \square

Remark 3.2. In [13], the inequality $\text{cat}(X^n) \leq \text{TC}_{n+1}(X)$ appears to have been overlooked. It is taken up in [1, Prop. 3.1], although by an apparently more involved argument than that used here.

We now prove our first main result enunciated in Section 1.

Proof of Theorem 1. Let $m: X \times X \rightarrow X$ denote the multiplication. Without loss of generality we may assume the unit is strict, i.e., $m(x, x_0) = x$ and $m(x_0, x) = x$ for $x \in X$, where $x_0 \in X$ denotes the basepoint. As is well-known by work of James [12], for any CW complex A , the set of pointed homotopy classes $[A, X]$ is an algebraic loop since X is connected (connectivity is implicitly used in James’s argument as it relies on Sugawara’s [16] showing that the shear map induces automorphisms on homotopy groups). In particular, the equation $a \cdot z = b$, for $a, b \in [A, X]$, has a unique solution $z \in [A, X]$. Here and in the remainder of this proof, we use “ \cdot ” to denote the product in $[A, X]$ induced by the multiplication. Write $\pi_1, \pi_2: X \times X \rightarrow X$ for the projections. In $[X \times X, X]$, we take $a = [\pi_1]$ and $b = [\pi_2]$, and we set $D: X \times X \rightarrow X$ to be the unique solution to the equation $[\pi_1] \cdot [D] = [\pi_2]$. That is, we have $\pi_1 \cdot D \sim \pi_2$. (If X were homotopy-associative with inverse, then here we could take $D(x, y) = x^{-1}y$.)

Now use the map $D: X \times X \rightarrow X$ to define a map $f_n: X^{n+1} \rightarrow X^n$ as $f_n(x, x_1, \dots, x_n) = (D(x, x_1), \dots, D(x, x_n))$. Form the pullback

$$(1) \quad \begin{array}{ccc} Q & \longrightarrow & \mathcal{P}X \\ q \downarrow & & \downarrow P_n \\ X^{n+1} & \xrightarrow{f_n} & X^n, \end{array}$$

so that we have

$$Q = \{(x, x_1, \dots, x_n, \alpha) \in X^{n+1} \times \mathcal{P}X \mid \alpha(0) = x_0, \alpha(t_i) = D(x, x_i), i = 1, \dots, n\},$$

and $q: Q \rightarrow X^{n+1}$ is just projection on the first $n + 1$ coordinates. Finally, define a map $\theta: Q \rightarrow \mathcal{P}X$ by setting $\theta((x, x_1, \dots, x_n, \alpha)) = m(x, \alpha)$, which gives a path in X evaluated as $m(x, \alpha)(t) = m(x, \alpha(t))$ for each $t \in I$. By definition, we have

$$P_{n+1} \circ \theta((x, x_1, \dots, x_n, \alpha)) = (m(x, x_0), m(x, D(x, x_1)), \dots, m(x, D(x, x_n))).$$

The first coordinate on the right-hand side here is x , since we are assuming a strict unit. Now write $p_j: X^{n+1} \rightarrow X$ for projection on the i th coordinate, and write $p_{1,j}: X^{n+1} \rightarrow X^2$ for the projection $p_{1,j} = (p_1, p_{j+1})$, so that $p_{1,j}(x, x_1, \dots, x_n) = (x, x_j)$. Then for $j = 1, \dots, n$, we may write the $(j + 1)$ st coordinate of the above expression as $m(x, D(x, x_j)) = (\pi_1 \cdot D) \circ p_{1,j}((x, x_1, \dots, x_n))$. Bearing in mind that $\pi_1 \cdot D \sim \pi_2$, we compute that

$$\begin{aligned} P_{n+1} \circ \theta &= (p_1, (\pi_1 \cdot D) \circ p_{1,1}, (\pi_1 \cdot D) \circ p_{1,2}, \dots, (\pi_1 \cdot D) \circ p_{1,n}) \circ q \\ &\sim (p_1, \pi_2 \circ p_{1,1}, \pi_2 \circ p_{1,2}, \dots, \pi_2 \circ p_{1,n}) \circ q \\ &= (p_1, p_2, p_3, \dots, p_{n+1}) \circ q = q. \end{aligned}$$

Thus we have a homotopy commutative diagram

$$(2) \quad \begin{array}{ccc} Q & \xrightarrow{\theta} & PX \\ & \searrow q & \swarrow P_{n+1} \\ & & X^{n+1}. \end{array}$$

From diagrams (1) and (2) it follows that $\text{secat}(P_{n+1}) \leq \text{secat}(q) \leq \text{secat}(P_n)$. The first of these is $\text{TC}_{n+1}(X)$ and the last of these is $\text{cat}(X^n)$ (since PX is contractible). Thus, for X an H -space we have $\text{TC}_{n+1}(X) \leq \text{cat}(X^n)$, and equality now follows from the general inequality $\text{cat}(X^n) \leq \text{TC}_{n+1}(X)$. \square

Examples 3.3. Note that $\text{cat}(G)$, and more so $\text{cat}(G^n)$, is known in relatively few cases in which G is an H -space. Exact values for $\text{cat}(G)$ are known for G a (special) unitary group, for G one of a small number of low rank symplectic or special orthogonal groups, and for G any rank-2 H -space or G one of certain rank-3 H -spaces (see [2, Sec. 4.4, Sec. 9.6] for a summary). We give some applications of Theorem 1 using some known results.

- (1) Theorem 1 applies to any compact Lie group, as was previously known. Thus we have, for instance, $\text{TC}(Sp(3)) = \text{cat}(Sp(3)) = 5$ as computed in [8], and $\text{TC}(Sp(9)) = 8$ as follows from [10].
- (2) Let $U(r)$ and $SU(r)$ denote the unitary group and special unitary group, respectively. It is known that $\text{cat}(U(r)) = r$ and $\text{cat}(SU(r)) = r - 1$ (see [15] or [2, Th. 9.47]). Although $\text{cat}(-)$ generally behaves subadditively with respect to products, we may see that $\text{cat}((U(r))^n) = nr$ and $\text{cat}((SU(r))^n) = n(r - 1)$. This is because, *in these cases*, we have a binding lower bound for $\text{cat}(-)$ given by (rational) cohomology, which equals the upper bound given by the product inequality. Thus it follows that we have $\text{TC}_{n+1}(U(r)) = nr$ and $\text{TC}_{n+1}(SU(r)) = n(r - 1)$.
- (3) The H -space S^7 is not homotopy-associative [11], and in particular is not a topological group. Here we do have the equality $\text{TC}_{n+1}(S^7) = n$ (see [4, Th. 8] for $n = 1$ and [13] for $n \geq 2$). But by taking products of S^7 with any H -space Y (Lie group or not), Theorem 1 leads to new equalities $\text{TC}_{n+1}(S^7 \times Y) = \text{cat}((S^7 \times Y)^n)$, where $S^7 \times Y$ is not a homotopy-associative H -space (and in particular is not a topological group). These equalities lead to many new cases in which $\text{TC}_{n+1}(-)$ may be determined. For example, we have $\text{cat}((S^7)^r) = r$; hence $\text{TC}_{n+1}((S^7)^r) = nr$ from Theorem 1.

- (4) Arguing as in Example (2), we have new equalities $TC_{n+1}(S^7 \times U(r)) = \text{cat}((S^7 \times U(r))^n) = n(r + 1)$ and $TC_{n+1}(S^7 \times SU(r)) = \text{cat}((S^7 \times SU(r))^n) = nr$.
- (5) For H -spaces which are not compact Lie groups, the L-S category is known only in very few cases. It is thus worth mentioning that it has been computed for all rank 2 H -spaces [3, Sec. 4]. In particular, for the Hilton-Roitberg criminal E_5 , or any other H -space in the genus of $Sp(2)$, whether homotopy-associative or not, one has $TC(E_5) = \text{cat}(E_5) = \text{cat}(Sp(2)) = 3$. Likewise the category of any space in the genus of $Sp(3)$ is 5 [9]; hence the topological complexity of any H -space in the genus of $Sp(3)$ is 5. Generally, the behaviour of the L-S category with respect to products of examples of this sort is not known. Determining higher topological complexity, even for G a rank-2 H -space, therefore remains a delicate problem.

Although in general both $\text{cat}(-)$ and $TC_{n+1}(-)$ behave subadditively with respect to products, there are situations in which they behave additively. Theorem 1 leads to the following observation.

Corollary 3.4. *Suppose X and Y are H -spaces. For each $n \geq 1$, we have a product equality $TC_{n+1}(X \times Y) = TC_{n+1}(X) + TC_{n+1}(Y)$ if and only if the identity $\text{cat}((X \times Y)^n) = \text{cat}(X^n) + \text{cat}(Y^n)$ holds.* □

Theorem 1 also leads to other interesting relations involving products and topological complexities, such as the following.

Corollary 3.5. *Let X be any CW H -space. Then for $n \geq 2$ and $k \geq 1$, we have*

$$TC_n(X^k) = TC_{k+1}(X^{n-1}).$$

Proof. Both are equal to $\text{cat}(X^{k(n-1)})$, by Theorem 1. □

Examples 3.6. [13, Prop. 5.1] gives the inequality $TC_n(T^2) \geq 2n - 2$, where $T^2 = S^1 \times S^1$ is the 2-torus.

(Note: The TC_n of [13] is the number of open sets in a cover, and is thus one greater than our TC_n .) We may improve on this inequality, as does [1, Cor. 3.13], with the formula

$$TC_n(T^k) = \text{cat}(T^{k(n-1)}) = k(n - 1),$$

for $n \geq 2$ and $k \geq 1$, with $T^k = (S^1)^k$ the k -torus. Thus, for example, we have identities $TC_3(S^1) = TC(T^2)$ (both equal 2), and $TC_4(T^2) = TC_3(T^3)$ (both equal 6). We also observe the sequences $\{TC_{n+1}(T^k)\}_{n \in \mathbb{N}} = \{kn\}_{n \in \mathbb{N}}$ as examples of the sequence Rudyak asks after in [13].

We end this section by sketching a variant of the proof we have given for Theorem 1. We feel the variant is worth mentioning as it uses the following intermediate result, of interest in its own right.

Proposition 3.7. *Let X be a connected, CW H -space. For $n \geq 1$, let $f_n: X^{n+1} \rightarrow X$ be the map defined in the proof of Theorem 1 using the map $D: X \times X \rightarrow X$. Then there is a homotopy pullback diagram*

$$\begin{array}{ccc}
 X & \longrightarrow & * \\
 \Delta_{n+1} \downarrow & & \downarrow \\
 X^{n+1} & \xrightarrow{f_n} & X^n.
 \end{array}$$

That is, the diagonal $\Delta_{n+1}: X \rightarrow X^{n+1}$ may be viewed as the fibre inclusion of the map $f_n: X^{n+1} \rightarrow X^n$.

Sketch of proof. One shows that the map $\theta: Q \rightarrow PX$, as in diagram (2) in the proof of Theorem 1, is a homotopy equivalence. Now by standard results on homotopy pullbacks, we may replace Q in the pullback diagram (1) by the homotopy equivalent space PX which, in turn, may be replaced by X , and the map q by P_{n+1} and, in turn, Δ_{n+1} . As usual, we may further replace PX by a point. \square

Proof (bis) of Theorem 1. From the homotopy pullback of Proposition 3.7, it follows that $\text{secat}(\Delta_{n+1}) \leq \text{secat}(* \rightarrow X^n)$, that is, that $\text{TC}_{n+1}(X) \leq \text{cat}(X^n)$. \square

4. SHEAR MAPS AND HOMOTOPY ACTIONS

In this section we generalize Theorem 1. Our first step in this direction is to generalize the condition that X be an H -space. Suppose we have a (right) action of a space Y on a space X , with strict (right) unit. That is, we have a map

$$A: X \times Y \rightarrow X$$

that satisfies $A(x, y_0) = x$, for all $x \in X$ and for $y_0 \in Y$ the basepoint. Write $\pi_1, \pi_2: X \times X \rightarrow X$ and $\pi^1: X \times Y \rightarrow X$ for the projections, preserving previous notation but also distinguishing the distinct domains by our notation. Then we may define a corresponding *shear map*

$$\text{sh}: X \times Y \rightarrow X \times X$$

by $\text{sh} = (\pi^1, A)$, so that $\text{sh}(x, y) = (x, A(x, y))$ for $(x, y) \in X \times Y$.

Definition 4.1. For an action $A: X \times Y \rightarrow X$ as above, we call a map $D: X \times X \rightarrow Y$ a (*homotopy*) *difference (map)* if it satisfies

$$A \circ (\pi_1, D) \sim \pi_2: X \times X \rightarrow X.$$

If an action admits a difference, we say that the action is *homotopy-transitive*.

Remark 4.2. In other words, a homotopy difference map satisfies the identity $A(x, D(x, x')) = x'$ up to homotopy; we think of D as continuously assigning, to each pair (x, x') , an element that acts on x so as to “translate it” to x' . If $X = Y = G$, a topological group, and if the action is the group multiplication, then we may set $D(x, y) = x^{-1}y$. We used the difference map $D: X \times X \rightarrow X$ in the proof of Theorem 1, in the context of an H -space X acting on itself by multiplication.

Lemma 4.3. *Let $A: X \times Y \rightarrow X$ be an action with strict unit and shear map $\text{sh} = (\pi^1, A): X \times Y \rightarrow X \times X$. The shear map admits a right homotopy inverse if and only if the action admits a difference $D: X \times X \rightarrow Y$.*

Proof. In fact, homotopy classes of right inverses to the shear map correspond bijectively to homotopy classes of differences under the identity $[\sigma] = [(\pi_1, D)]$. Observe that we may write $1_{X \times X} = (\pi_1, \pi_2): X \times X \rightarrow X \times X$. Now suppose we are given a right inverse $\sigma = (\sigma_1, \sigma_2)$ of $\text{sh}: X \times X \rightarrow X \times Y$. Define $D = \sigma_2: X \times X \rightarrow Y$. Then we have

$$(\pi_1, \pi_2) = 1_{X \times X} \sim \text{sh} \circ \sigma = (\pi^1, A) \circ (\sigma_1, D) \sim (\sigma_1, A \circ (\sigma_1, D)).$$

Projecting onto first coordinates gives $\pi_1 \sim \sigma_1$, and then projecting onto second coordinates gives $\pi_2 \sim A \circ (\sigma_1, D) \sim A \circ (\pi_1, D)$, so that D is a difference. Evidently, homotopic right inverses result in homotopic differences.

Conversely, suppose given a difference $D: X \times X \rightarrow Y$, and define $\sigma = (\pi_1, D): X \times X \rightarrow X \times Y$. Then we have $\text{sh} \circ \sigma = (\pi^1, A) \circ (\pi_1, D) = (\pi_1, A \circ (\pi_1, D)) \sim (\pi_1, \pi_2) = 1_{X \times X}$. Once again, it is evident that homotopic differences result in homotopic right inverses. \square

If an action $A: X \times Y \rightarrow X$ is homotopy-transitive, we may repeat the steps of the proof of Theorem 1, suitably adapted, to obtain inequalities $\text{TC}_{n+1}(X) \leq \text{cat}(Y^n)$ for each $n \geq 1$, and in particular the inequality $\text{TC}(X) \leq \text{cat}(Y)$. In principle, this appears to be a satisfactory generalization of Theorem 1. However, the following observation reveals that these upper bounds would actually be weaker than that given by Theorem 1 in all cases in which they apply.

Lemma 4.4. *Let $A: X \times Y \rightarrow X$ be a homotopy-transitive action. Then*

- (1) X is a retract of Y , and
- (2) X is an H -space.

Proof. Let $\sigma = (\pi_1, D)$ be a right inverse of the shear map (π^1, A) , so that $D: X \times X \rightarrow Y$ is a difference that satisfies $A \circ (\pi_1, D) \sim \pi_2$. Then define $j = D \circ i_2: X \rightarrow Y$. Also, let $a = A \circ i_2: Y \rightarrow X$ be the usual orbit map of the action. We observe that $i_2 \circ D \circ i_2 \sim \sigma \circ i_2: X \rightarrow X \times X \rightarrow X \times Y$, and hence we have

$$\begin{aligned} a \circ j &\sim (A \circ i_2) \circ (D \circ i_2) = A \circ (i_2 \circ D \circ i_2) \\ &\sim (\pi_2 \circ \text{sh}) \circ (\sigma \circ i_2) = \pi_2 \circ (\text{sh} \circ \sigma) \circ i_2 \sim \pi_2 \circ i_2 = 1_X. \end{aligned}$$

Thus, the orbit map $a: Y \rightarrow X$ is a retraction of Y onto X .

For (2), we define $m = A \circ (1 \times j): X \times X \rightarrow X$. For m to be a multiplication, we only need check that it satisfies the two-sided unit condition: $m \circ i_1 \sim 1_X \sim m \circ i_2: X \rightarrow X$. The second identity is what we have just shown: $a \circ j \sim 1_X$. The first follows immediately from the assumptions that j is a based map (which we may assume, without loss of generality, if spaces are well-pointed) and that the action has a strict unit. \square

To generalize Theorem 1, then, we go further and also relax the requirement of a section (right homotopy inverse) of the shear map. We use instead local sections, exactly as in the notion of sectional category. This leads to local differences, which may be used in the same way as their global counterparts, with suitable adaptations of the argument. The upshot is the second main result enunciated in Section 1, whose proof we now give. Notice that the result places no assumption on the action: although the conclusion may not be particularly strong, it has the advantage of applying very generally.

Proof of Theorem 2. Suppose that $\text{secat}(\text{sh}_{n+1}) = m$, so that we have $m + 1$ open sets $\{U_0, \dots, U_m\}$ which cover X^{n+1} and for each of which we have a local section of sh_{n+1} . Fix an i , and write the corresponding local section of sh_{n+1} as $\sigma = (\sigma_1, D_1, \dots, D_n): U_i \rightarrow X \times Y^n$ to define maps $D_j: U_i \rightarrow Y$. As in Section 3 and the enunciation of Theorem 2 in Section 1, write $p_k: X^{n+1} \rightarrow X$, $p^1: X \times Y^n \rightarrow X$,

and $p^{k \geq 2}: X \times Y^n \rightarrow Y$ for the projections. For the inclusion $\text{incl}: U_i \rightarrow X^{n+1}$,

$$\begin{aligned} \text{incl} \sim \text{sh}_{n+1} \circ \sigma &= (p^1, A \circ (p^1, p^2), \dots, A \circ (p^1, p^{n+1})) \circ (\sigma_1, D_1, \dots, D_n) \\ &= (\sigma_1, A \circ (\sigma_1, D_1), \dots, A \circ (\sigma_1, D_n)). \end{aligned}$$

Equating coordinates yields $\sigma_1 \sim p_1: U_i \rightarrow X$, and then $A \circ (p_1, D_j) \sim p_{j+1}: U_i \rightarrow X$ for each $j = 1, \dots, n$. In this way, the D_j are thus *local differences* on U_i . Write $D^i = (D_1, \dots, D_n): U_i \rightarrow Y^n$, and form the pullback

$$(3) \quad \begin{array}{ccc} Q_i & \longrightarrow & \mathcal{P}Y \\ q_i \downarrow & & \downarrow \mathcal{P}_n \\ U_i & \xrightarrow{D^i} & Y^n, \end{array}$$

so that

$$Q_i = \{(x, x_1, \dots, x_n, \alpha) \in U_i \times \mathcal{P}Y \mid \alpha(0) = y_0, \alpha(t_j) = D_j(x, x_1, \dots, x_n)\},$$

and $q_i: Q_i \rightarrow U_i$ is just projection on the first $n + 1$ coordinates. Finally, define a map $\theta_i: Q_i \rightarrow PX$ by setting $\theta_i((x, x_1, \dots, x_n, \alpha)) = A(x, \alpha)$, which gives a path in X evaluated as $A(x, \alpha)(t) = A(x, \alpha(t))$ for each $t \in I$. Write the n -tuple (x_1, \dots, x_n) as \mathbf{x} , and $(x, x_1, \dots, x_n, \alpha)$ as (x, \mathbf{x}, α) . Then composing θ_i and $P_{n+1}: PX \rightarrow X^{n+1}$ gives

$$\begin{aligned} P_{n+1} \circ \theta_i(x, \mathbf{x}, \alpha) &= (A(x, \alpha(0)), A(x, \alpha(t_1)), \dots, A(x, \alpha(t_n))) \\ &= (A(x, y_0), A(x, D_1(x, \mathbf{x})), \dots, A(x, D_n(x, \mathbf{x}))) \\ &= (p_1, A \circ (p_1, D_1), \dots, A(p_1, D_n)) \circ q_i(x, \mathbf{x}, \alpha), \end{aligned}$$

which gives $P_{n+1} \circ \theta_i \sim \text{incl} \circ q_i$ as maps into X^{n+1} , from the argument above diagram (3). Since P_{n+1} is a fibration, we may replace θ_i by a homotopic map such that $P_{n+1} \circ \theta_i = \text{incl} \circ q_i$, and thus we have a commutative diagram

$$\begin{array}{ccc} Q_i & \xrightarrow{\theta_i} & PX \\ q_i \downarrow & & \downarrow P_{n+1} \\ U_i & \xrightarrow{\text{incl}} & X^{n+1}. \end{array}$$

From this diagram, local sections of q_i compose with θ_i to give local sections of P_{n+1} . Recall that the steps taken thus far were for a typical U_i in the original $(m + 1)$ -fold cover of X^{n+1} . It follows that we have an upper bound

$$\text{TC}_{n+1}(X) = \text{secat}(P_{n+1}) \leq \sum_{i=0}^m (\text{secat}(q_i) + 1) - 1.$$

From pullback diagram (3), for each i we have $\text{secat}(q_i) \leq \text{cat}(Y^n)$. Hence we have

$$\text{TC}_{n+1}(X) \leq (\text{secat}(\text{sh}_{n+1}) + 1)(\text{cat}(Y^n) + 1) - 1. \quad \square$$

Observe that Theorem 1 is indeed a special case of Theorem 2: in the former case, we have $Y = X$ acting transitively on itself and $\text{secat}(\text{sh}_{n+1}) = 0$ (the shear map admits a section).

Remark 4.5. An in-principle finer upper bound on $\text{TC}_{n+1}(X)$ is obtained by returning to diagram (3) in the proof above and observing that, for each i , we have $\text{secat}(q_i) = \text{cat}(D^i)$. Here, $\text{cat}(D^i)$ denotes the *L-S category of the map* D^i ; the equality follows from standard results about $\text{secat}(-)$ [2, Prop. 9.18]. This yields the upper bound $\text{TC}_{n+1}(X) \leq \sum_{i=0}^m (\text{cat}(D^i) + 1) - 1$.

5. OPEN QUESTIONS

5.1. Is it possible to characterize spaces X for which $\text{TC}_{n+1}(X)$ takes either its minimum value of $\text{cat}(X^n)$ or its maximum value of $\text{cat}(X^{n+1})$? We have established that the minimum value is taken by H -spaces. Examples at the other extreme include even-dimensional spheres, since we have $\text{TC}_{n+1}(S^{2r}) = n + 1$ [13, Sec. 4], and $\text{cat}((S^{2r})^{n+1}) = n + 1$.

5.2. Examples 3.6 prompt the following: For powers X^k of a space X , (when) is there a relation (such as equality) between $\text{TC}_n(X^k)$ and $\text{TC}_{k+1}(X^{n-1})$?

Regarding this question, notice that the standard inequalities

$$\text{cat}(X^{k(n-1)}) \leq \text{TC}_n(X^k) \leq \text{cat}(X^{kn})$$

and

$$\text{cat}(X^{k(n-1)}) \leq \text{TC}_{k+1}(X^{n-1}) \leq \text{cat}(X^{(n-1)(k+1)})$$

give a common lower bound, but different upper bounds. Relations of a similar nature are given in [1, Cor. 3.5, Cor. 3.6].

5.3. From Proposition 3.7 (cf. Remark 4.5) it follows that, for X an H -space, we have that $\text{TC}_{n+1}(X) = \text{cat}(f_n)$ (as well as being equal to $\text{cat}(X^n)$). Are there other situations in which $\text{TC}_{n+1}(X)$ may be identified as the L-S category of some auxiliary map or space?

5.4. We mention briefly another invariant of the L-S type, namely *relative category* (see [2, Sec. 7.2]). This is an invariant of a map $p: E \rightarrow B$, which we denote by $\text{relcat}(p)$. We omit the definition here, but record the general inequalities $\text{secat}(p) \leq \text{relcat}(p) \leq \text{cat}(B) + 1$. This invariant would seem of interest with regard to $\text{TC}(-)$ because of the resulting inequalities $\text{TC}_n(X) \leq \text{relcat}(\Delta_n) \leq \text{cat}(X^n) + 1$. Is it possible to identify situations in which $\text{relcat}(\Delta_n)$ gives a better upper bound for $\text{TC}_n(X)$ than the usual $\text{cat}(X^n)$?

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