Homotopie et Homologie
Exercise Set 9

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1. Let A be a subspace of X, and let $x_0 \in A$. Let P_i denote the homotopy fiber of the inclusion $i : A \hookrightarrow X$ (cf. Definition 2, Exercise Set 8). Show that
$$\pi_n(X, A) \cong \pi_{n-1} P_i.$$

2. Let $\{x_0\} \subseteq A \subseteq B \subseteq X$ be a sequence of subspace inclusions. Show that there is an exact sequence
$$\cdots \to \pi_n(B, A) \to \pi_n(X, A) \to \pi_n(X, B) \xrightarrow{\partial_n} \pi_{n-1}(B, A) \to \cdots .$$

   Hint: The *connecting homomorphism* ∂_n is equal to the composite
$$\pi_n(X, B) \to \pi_{n-1} B \to \pi_{n-1}(B, A).$$

3. Let $j : A \hookrightarrow X$ denote the inclusion of a closed subspace.
   (a) Prove that if j is a cofibration, then so is
$$j \times Id_Z : A \times Z \to X \times Z$$
   for all spaces Z.
   (b) Prove that if j is a cofibration, then for all maps $f : A \to Y$, the induced inclusion
$$Y \hookrightarrow Y \cup_f X = Y \coprod X/\sim,$$
   where $a \sim f(a)$ for all $a \in A$, is also a cofibration, i.e., cofibrations are preserved under pushout. In particular, for all maps $f : X \to Y$, the inclusion $i_f : Y \hookrightarrow C_f$ is a cofibration.

4. Prove that if $i : A \hookrightarrow B$ and $j : B \hookrightarrow X$ are cofibrations, then so is $j \circ i : A \hookrightarrow X$. Prove more generally that if
$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots$$

are inclusions of closed subspaces that are cofibrations, then the inclusion $A_0 \hookrightarrow \bigcup_{n\geq 0} A_n$ is a cofibration, if the topology on $\bigcup_{n\geq 0} A_n$ satisfies:

$$C \subseteq \bigcup_{n\geq 0} A_n \text{ closed } \Leftrightarrow C \cap A_n \text{ closed in } A_n \forall n.$$