Homotopie et Homologie
Exercise Set 8

07.11.2013 and 14.11.2013

The goal of this series of exercises is to develop and study the dual Barrat-Puppe sequence. We begin by defining the constructions that are analogous to the cone and mapping cone constructions.

Definition 1. The *based path space* on a pointed space (Y, y_0), denoted PY, is the *pullback* of

$$\{y_0\} \hookrightarrow Y \xleftarrow{ev_0} \mathrm{Map}(I, Y),$$

where $ev_0(\lambda) = \lambda(0)$, i.e., $PY = \big\{\lambda \in \mathrm{Map}(I, Y) \mid \lambda(0) = y_0\big\}$. Let $e : PY \to Y$ denote the map given by $e(\lambda) = \lambda(1)$.

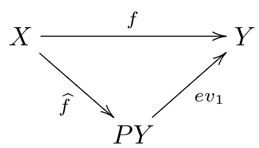
Definition 2. The *homotopy fiber* of a map $f : X \to Y$, denoted P_f, is the *pullback* of

$$X \xrightarrow{f} Y \xleftarrow{e} PY,$$

i.e., $P_f = \big\{(x, \lambda) \in X \times PY \mid \lambda(1) = x\big\}$, which is a subspace of $X \times PY$. Let $q_f : P_f \to X : (x, \lambda) \to x$.

1. The based path space and homotopy fiber constructions allow us to formulate interesting characterizations of nullhomotopic maps.

   (a) Prove that a pointed map $f : (X, x_0) \to (Y, y_0)$ is nullhomotopic if and only if there is a pointed map $\widehat{f} : (X, x_0) \to (PY, c_{y_0})$ such that

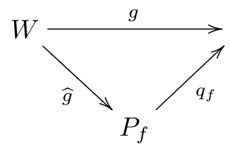
   

   commutes.

   (b) Let $(W, w_0) \xrightarrow{g} (X, x_0) \xrightarrow{f} (Y, y_0)$ be pointed maps. Prove that $f \circ g$ is nullhomotopic if and only if there exists a pointed map

   $$\widehat{g} : (W, w_0) \to \big(P_f, (x_0, c_{y_0})\big)$$

such that



commutes.

2. Compute the homotopy fibers of the inclusion of the basepoint $\{y_0\}$ into Y, of $e : PY \to Y$ and of $q_f : P_f \to X$, for any pointed map $f : (X, x_0) \to (Y, y_0)$. Show that $\Omega P_f \simeq_* P_{\Omega f}$ as well.

3. Prove that for any pointed map $f : (X, x_0) \to (Y, y_0)$ (respectively, H-morphism $f : (X, x_0, \mu) \to (Y, y_0, \nu)$), the sequence $P_f \xrightarrow{q_f} X \xrightarrow{f} Y$ is h-exact, i.e., for any pointed space (W, w_0), the sequence of set maps (respectively, of homomorphisms)

$$[W, P_f]_* \xrightarrow{(q_f)_*} [W, X]_* \xrightarrow{f_*} [W, Y]_*$$

is exact.

4. Given a pointed map $f : (X, x_0) \to (Y, y_0)$, explain how to obtain a sequence of pointed maps

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to P_{\Omega f} \to \Omega X \to \Omega Y \to P_f \to X \to Y$$

(basepoints suppressed) by iterating the homotopy fiber construction: the *dual Barratt-Puppe sequence.* Conclude, using exercise 3, that for any pointed space (W, w_0), the sequence

$$\cdots \to [W, P_{\Omega f}]_* \to [W, \Omega X]_* \to [W, \Omega Y]_* \to [W, P_f]_* \to [W, X]_* \to [W, Y]_*$$

is exact and therefore that in particular there is an exact sequence

$$\cdots \to \pi_n P_f \to \pi_n X \to \pi_n Y \to \pi_{n-1} P_f \to \cdots \to \pi_1 Y \to \pi_0 P_f \to \pi_0 X \to \pi_0 Y.$$

This is the *long exact sequence in homotopy* of the *fiber sequence*

$$P_f \xrightarrow{q_f} X \xrightarrow{f} Y.$$

5. Prove that $\pi_n S^1 = 0$ for all $n \geq 2$ and then that $\pi_n S^3 \cong \pi_n S^2$ for all $n \geq 3$.

Hint: Apply exercise 4 first to $\exp : \mathbb{R} \to S^1$ and then to the *Hopf map*

$$\eta : S^3 \to S^2 : (x_1, x_2, x_3, x_4) \mapsto (x_1x_3 + x_2x_4, x_2x_3 - x_1x_4, x_1^2 + x_2^2 - x_3^2 - x_4^2).$$