Homotopie et Homologie
Exercise Set 7

31.10.2013

1. Prove that the fundamental group of any H-space (not necessarily homotopy associative!) is abelian.

2. Let $f : (X, x_0) \to (X', x_0')$ be a pointed, continuous map. Explain how to extend the based loop space and suspension constructions to define functors
$$\Omega : \mathbf{Top}_* \to \textbf{H-spaces}$$
and
$$\Sigma : \mathbf{Top}_* \to \textbf{co-H-spaces}.$$

3. For any pointed space (X, x_0), define pointed, continuous maps
$$\iota_{(X,x_0)} : (X, x_0) \to \Omega\Sigma(X, x_0)$$
and
$$\rho_{(X,x_0)} : \Sigma\Omega(X, x_0) \to (X, x_0)$$
such that for all pointed, continuous maps $f : (X, x_0) \to (X', x_0')$,
$$\Omega\Sigma f \circ \iota_{(X,x_0)} = \iota_{(X',x_0')} \circ f : (X, x_0) \to \Omega\Sigma(X', x_0'),$$
and
$$f \circ \rho_{(X,x_0)} = \rho_{(X',x_0')} \circ \Sigma\Omega f : \Sigma\Omega(X, x_0) \to (X', x_0'),$$
i.e., ι is a natural transformation from $Id_{\mathbf{Top}_*}$ to $\Omega\Sigma$, while ρ is a natural transformation from $\Sigma\Omega$ to $Id_{\mathbf{Top}_*}$.
Moreover,
$$\alpha : \mathrm{Map}\big(\Sigma(X, x_0), (X', x_0')\big) \to \mathrm{Map}\big((X, x_0), \Omega(X', x_0')\big) : g \mapsto \Omega g \circ \iota_{(X,x_0)}$$
and
$$\beta : \mathrm{Map}\big((X, x_0), \Omega(X', x_0')\big) \to \mathrm{Map}\big(\Sigma(X, x_0), (X', x_0')\big) : g \mapsto \rho_{(X',x_0')} \circ \Sigma g$$
are mutually inverse H-isomorphisms (i.e., homeomorphisms that are H-morphisms).