Homotopie et Homologie
Exercise Set 6

24.10.2013

1. Suppose that $\big[-,(Y,y_0)\big]_*:\mathbf{Top}_*\to\mathbf{Gr}$. Let $\mu:Y\times Y\to Y$ be a based continuous map such that $[\mu]_*=[pr_1]_*\cdot[pr_2]_*$. Prove that

$$\mu\circ(\mu\times Id_Y)\simeq_*\mu\circ(Id_Y\times\mu).$$

2. Prove that if (Y,y_0,η,μ,σ) is an H-groupe, then

$$\big[-,(Y,y_0)\big]_*:\mathbf{Top}_*\to\mathbf{Gr}.$$

3. Let (X,x_0,ψ) be a co-H-space. Show that if $f:(X,x_0)\to(X',x_0')$ is a based homotopy equivalence, then X' admits a co-H-space structure with respect to which f is a co-H-morphism. Show that, moreover, if X is actually a co-H-group, then so is X'. In other words, "to be a co-H-space (respectively, co-H-group)" is a homotopy invariant notion.

4. Let (X,x_0,ψ) be a co-H-space, and let (Y,y_0) be any based space. Show that $\mathrm{Map}\big((X,x_0),(Y,y_0)\big)$ admits an H-space structure, which is natural in the sense that if $f:(Y,y_0)\to(Z,z_0)$ is a based, continuous map, then the induced map

$$f_*:\mathrm{Map}\big((X,x_0),(Y,y_0)\big)\to\mathrm{Map}\big((X,x_0),(Z,z_0)\big)$$

is an H-morphism.

5. It is easy to see that there are interesting H-groups with strictly associative multiplication, strict units and strict inverses: any topological group (S^1, $GL(n,\mathbb{R})$, $O(n)$, $U(n)$,...) is an example of such. Show that, on the other hand, there are no nontrivial, strict co-H-spaces.