

Homotopie et Homologie

Exercise Set 4

10.10.2013

- (Wedges and products: application of Mayer-Vietoris) Let G be an abelian group.
 - Calculate $H_*(S^1 \vee S^1; G)$. Here, $S^1 \vee S^1 = S^1 \amalg S^1 / \sim$, where the equivalence relation identifies (glues together) only the basepoints of the two copies of S^1 .
 - Calculate $H_*(S^1 \times S^1; G)$.
- (Moore spaces: application of Mayer-Vietoris) For all $d \in \mathbb{Z}$, let

$$\lambda_d : S^1 \rightarrow S^1 : e^{i\theta} \mapsto e^{id\theta}.$$

As we will see later in the semester, the induced homomorphism

$$H_1(\lambda_d; \mathbb{Z}) : H_1(S^1; \mathbb{Z}) \rightarrow H_1(S^1; \mathbb{Z})$$

is simply given by multiplication by d .

- Prove that for every integer $d \geq 1$, there exists a space $M(\mathbb{Z}/d\mathbb{Z}, 1)$ such that

$$H_m(M(\mathbb{Z}/d\mathbb{Z}, 1); \mathbb{Z}) = \begin{cases} \mathbb{Z} & : m = 0 \\ \mathbb{Z}/d\mathbb{Z} & : m = 1 \\ 0 & : m > 1. \end{cases}$$

- Prove that for every pair of integers $d, n \geq 1$, there exists a space $M(\mathbb{Z}/d\mathbb{Z}, n)$ such that

$$H_m(M(\mathbb{Z}/d\mathbb{Z}, n); \mathbb{Z}) = \begin{cases} \mathbb{Z} & : m = 0 \\ \mathbb{Z}/d\mathbb{Z} & : m = n \\ 0 & : m \neq n. \end{cases}$$

The space $M(\mathbb{Z}/d\mathbb{Z}, n)$ is called a *Moore space of type* $(\mathbb{Z}/d\mathbb{Z}, n)$.

- (c) Let $A_* = (A_m)_{m \geq 1}$ be a sequence of finitely generated abelian groups. Prove that there is a topological space X such that

$$H_m(X; \mathbb{Z}) = A_m \quad \forall m \geq 1.$$

Let E_* denote a homology theory on \mathbf{Top}_{rel} . The next three exercises concern the sequence of homomorphisms

$$\cdots \rightarrow E_n(A, B) \xrightarrow{E_n(i, \text{Id}_B)} E_n(X, B) \xrightarrow{E_n(\text{Id}_X, j)} E_n(X, A) \xrightarrow{\delta_n} E_{n-1}(A, B) \rightarrow \cdots,$$

where $B \xrightarrow{j} A \xrightarrow{i} X$ is a sequence of inclusions of subspaces, and δ_n is the composite

$$E_n(X, A) \xrightarrow{(\partial_n)(X, A)} E_{n-1}(A, \emptyset) \xrightarrow{E_{n-1}(\text{Id}_A, u_B)} E_n(A, B).$$

3. Prove that $\text{Im } E_n(i, \text{Id}_B) = \ker_n(\text{Id}_X, j)$.
4. Prove that $\text{Im } E_n(\text{Id}_X, j) = \ker \delta_n$.
5. Prove that $\text{Im } \delta_n = \ker E_{n-1}(i, \text{Id}_B)$.