Homotopie et Homologie
Exercise Set 3

03.10.2013

1. Let E_* be a homology theory on Top_{rel}. Let $X_n = \{1, ..., n\}$, endowed with the discrete topology.
   (a) Calculate $E_*(X_n, \emptyset)$ in terms of $E_*(X_1, \emptyset)$.
   (b) Calculate $E_*(X_n, X_k)$ in terms of $E_*(X_1, \emptyset)$, for all $k \leq n$.
   (c) For all $n \geq 1$, let $Y_n = D^1 \coprod \cdots \coprod D^n$, where D^k is the disk of dimension k. Calculate $E_*(Y_n, Y_k)$ in terms of $E_*(X_1, \emptyset)$, for all $k \leq n$.

2. (The 5-Lemma) Let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

   be a commutative diagram in Ab with exact rows. Show that if

   - α_2 and α_4 are isomorphisms,
   - α_1 is surjective, and
   - α_5 is injective,

   then α_3 is an isomorphism.

3. Let $\tau_* : E_* \to E'_*$ be a morphism of homology theories on Top_{rel}, i.e., $\tau_* = \{\tau_n \mid n \geq 0\}$, where, for every n, $\tau_n : E_n \to E'_n$ is a natural transformation such that $\tau_{n-1}\partial_n = \partial'_n \tau_{n-1}$.

   Apply the 5-Lemma to proving that if

$$(\tau_n)_{(X,\emptyset)} : E_n(X, \emptyset) \to E'_n(X, \emptyset)$$

   is an isomorphism for all topological spaces X, then

$$(\tau_n)_{(X,A)} : E_n(X, A) \to E'_n(X, A)$$

   is an isomorphism for all pairs (X, A).

4. (Split exact sequences) A short exact sequence

$$0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0 \tag{0.1}$$

in Ab is *split* if p *admits a section*, i.e., if there exists a homomorphism $s : C \to B$ such that $p \circ s = \mathrm{Id}_C$.

   (a) Show that the sequence (0.1) is split if and only if j *admits a retraction*, i.e., there exists a homomorphism $r : B \to A$ such that $r \circ j = \mathrm{Id}_A$.

   (b) Show that the sequence (0.1) is split if and only if there is an isomorphism $\varphi : A \oplus C \xrightarrow{\cong} B$ such that $\varphi \circ \iota = j$ and $p \circ \varphi = \pi$, where $\iota : A \hookrightarrow A \oplus C$ and $\pi : A \oplus C \to C$ are the natural inclusion and projection, respectively.

   (c) Give two examples of nonsplit short exact sequences.