Homotopie et Homologie
Exercise Set 2

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The theme of this exercise set is the famous **Seifert-van Kampen Theorem**, a very useful tool for computing the fundamental group of a based space in terms of the fundamental groups of subspaces whose union is the whole space. Similar decomposition results hold for many of the most common homotopy invariants.

1. (A necessary algebraic notion) A *free product* (also known as the *co-product*) of two groups G_1 and G_2 consists of a group G and two homomorphisms $\iota_1 : G_1 \to G$ and $\iota_2 : G_2 \to G$ satisfying the following *universal property*: for any pair of homomorphisms $\varphi_1 : G_1 \to H$ and $\varphi_2 : G_2 \to H$, there is a **unique** homomorphism $\varphi : G \to H$ such that $\varphi \circ \iota_k = \varphi_k : G_k \to G$ for $k = 1, 2$.

   (a) Show that if the free product of two groups is unique up to isomorphism, i.e., if (G, ι_1, ι_2) and (G', ι_1', ι_2') both satisfy the universal property above for a pair of groups G_1 and G_2, then $G \cong G'$.

   (b) Prove the existence of the free product of any pair of groups.

   (**Hint:** Construct the group G explicitly in terms of *words*, the letters of which are elements of G_1 and G_2, then check that the universal property holds.)

   It now makes sense to introduce the notation $(G_1 * G_2, \iota_1, \iota_2)$ for *the* free product of G_1 and G_2 and $\varphi_1 * \varphi_2 : G_1 * G_2 \to H$ for the homomorphism satisfying $(\varphi_1 * \varphi_2) \circ \iota_k = \varphi_k$ for $k = 1, 2$.

   (c) For any group G, compute $G * \{e\}$, where $\{e\}$ denotes the trivial group.

   (d) Provide a description of $\mathbb{Z} * \mathbb{Z}$ in terms of paths in the lattice $\mathbb{Z} \times \mathbb{Z}$.

2. (Seifert-van Kampen) Let X be a topological space, and let X_1 and X_2 be open subspaces of X such that $X = X_1 \cup X_2$, $X_1 \cap X_2 \neq \emptyset$, and X_1, X_2 and $X_1 \cap X_2$ are all path connected. Let $j_1 : X_1 \hookrightarrow X$, $j_2 : X_2 \hookrightarrow X$, $i_1 : X_1 \cap X_2 \hookrightarrow X_1$ and $i_2 : X_1 \cap X_2 \hookrightarrow X_2$ denote the inclusion maps. Let $x_0 \in X_1 \cap X_2$.

(a) Prove that the homomorphism

$$\pi_1 j_1 * \pi_1 j_2 : \pi_1(X_1, x_0) * \pi_1(X_2, x_0) \to \pi_1(X, x_0)$$

is surjective.

(b) Let N denote the smallest normal subgroup of $\pi_1(X_1, x_0) * \pi_1(X_2, x_0)$ that contains all words of the form $\pi_1 i_1\big([\lambda]_*\big)\pi_1 i_2\big([\lambda]_*\big)^{-1}$. Prove that $N < \ker(\varphi_1 * \varphi_2)$.

Remark. It is also true, but harder to prove, that $\ker(\varphi_1 * \varphi_2) < N$, whence

$$\pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_1(X_2, x_0)/N$$

under the hypotheses above.

3. (Calculations using Seifert-van Kampen)

(a) Let $n \geq 2$. Prove that for any choice of basepoint z_0 in S^n, $\pi_1(S^n, z_0)$ is the trivial group. Why doesn't the proof work when $n = 1$?

(b) Let $S^1 \vee S^1$ denote the *wedge* of two copies of S^1, i.e,

$$S^1 \vee S^1 = \{(z, z') \in S^1 \times S^1 \mid z = z_0 \text{ or } z' = z_0\}.$$

Prove that $\pi_1\big(S^1 \vee S^1, (z_0, z_0)\big) \cong \mathbb{Z} * \mathbb{Z}$.

4. (Realizing cyclic groups as fundamental groups) For any positive integer n, explain how to construct a path-connected space X_n from a circle and a disk such that $\pi_1(X_n, x_0) \cong \mathbb{Z}/n\mathbb{Z}$. Show that X_2 is homeomorphic to $\mathbb{R}P^2$, the *real projective plane*, which is usually defined as the quotient of D^2 by the relation that identifies antipodal points on its boundary.