Homotopie et Homologie
Exercise Set 11

05.12.2013

1. Let $\beta : (\widetilde{X}, \widetilde{A}) \to (X, A)$ and $\gamma : (\widetilde{Y}, \widetilde{B}) \to (Y, B)$ be CW-approximations.

   (a) Show that for every continuous map of pairs $f : (X, A) \to (Y, B)$, there is a continuous map of pairs $\tilde{f} : (\widetilde{X}, \widetilde{A}) \to (\widetilde{Y}, \widetilde{B})$ such that $\gamma \circ \tilde{f} \simeq f \circ \beta$.

   (b) Show moreover that if $\tilde{f}' : (\widetilde{X}, \widetilde{A}) \to (\widetilde{Y}, \widetilde{B})$ is such that $\gamma \circ \tilde{f}' \simeq f \circ \beta$, then $\tilde{f} \simeq \tilde{f}'$, i.e., the lift of f to the CW-approximations is unique up to homotopy.

   Hint 1. Use Whitehead's Theorem!

2. We see in this exercise how to construct a canonical CW-approximation to a space X.

   For any $n \in \mathbb{N}$, let

$$\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \ \forall i\},$$

   and let

$$\partial^i : \Delta^{n-1} \to \Delta^n : (t_0, ..., t_{n-1}) \mapsto (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$$

   and

$$\sigma^i : \Delta^{n+1} \to \Delta^n : (t_0, ..., t_{n+1}) \mapsto (t_0, ..., t_i + t_{i+1}, ..., t_{n+1})$$

   for all $0 \leq i \leq n$.

   For any space X, let

$$S_n(X) = \{\varphi : \Delta^n \to X \mid \sigma \text{ continuous }\},$$

   seen as a discrete topological space, and

$$\Gamma X = \coprod_{n \geq 0} S_n(X) \times \Delta^n / \sim,$$

where

$$(\varphi \circ \sigma^i, \mathbf{t}) \sim \big(\varphi, \sigma^i(\mathbf{t})\big) \text{ and } (\varphi \circ \partial^i, \mathbf{s}) \sim \big(\varphi, \partial^i(\mathbf{s})\big)$$

for all $\varphi \in S_n(X)$, $\mathbf{t} \in \Delta^{n+1}$, $\mathbf{s} \in \Delta^{n-1}$, $0 \leq i \leq n$ and $n \geq 0$. Endow ΓX with the obvious quotient topology.

(a) Show that there is a CW-structure on ΓX such that the quotient map $\coprod_{n\geq 0} S_n(X) \times \Delta^n \to \Gamma X$ is a cellular map.

(b) Show that the evaluation maps $ev : S_n(X) \times \Delta^n \to X : (\varphi, \mathbf{t}) \mapsto \varphi(\mathbf{t})$ together give rise to a continuous map $\varepsilon_X : \Gamma X \to X$.

(c) Show that any map $f : X \to Y$ gives rise to a map $\Gamma f : \Gamma X \to \Gamma Y$ such that

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Remark 2. The map ε_X behaves very nicely: it satisfies a universal property (which we do not spell out here), and it is always a weak equivalence!

3. Let G be a group, endowed with the discrete topology. Let

$$BG = \coprod_{n\geq 0} G^{\times n} \times \Delta^n / \sim,$$

where $G^{\times 0} = \{e\}$, and

$$\big((a_1, ..., a_i, e, a_{i+1}, ..., a_n), \mathbf{t}\big) \sim \big((a_1, ..., a_{i-1}, a_i, ..., a_n), \sigma^i(\mathbf{t})\big),$$

for all $\mathbf{t} \in \Delta^{n+1}$ and $0 \leq i \leq n$, while for all $\mathbf{s} \in \Delta^{n-1}$,

$$\big((a_1, ..., a_i a_{i+1}, ..., a_n), \mathbf{s}\big) \sim \big((a_1, ..., a_i, a_{i+1}, ..., a_n), \partial^i(\mathbf{s})\big),$$

for all $0 < i < n$, while

$$\big((a_2, ..., a_n), \mathbf{s}\big) \sim \big((a_1, ..., a_n), \partial^0(\mathbf{s})\big) \text{ and } \big((a_1, ..., a_{n-1}), \mathbf{s}\big) \sim \big((a_1, ..., a_n), \partial^n(\mathbf{s})\big).$$

Endow BG with the obvious quotient topology.

Using the Cellular Approximation Theorem, prove that

$$\pi_1(BG, [e, 1]) \cong G.$$

Remark 3. The space BG is (a model for) the *classifying space* of G. One can actually prove that $\pi_k(BG, [e, 1]) = 0$ for all $k > 1$. In particular, $B\mathbb{Z} \simeq S^1$.