Homotopie et Homologie
Exercise Set 10

28.11.2013

1. Let X be a CW-complex. Prove the following properties of the topology on X.

   (a) The n-skeleton X_n of X is closed in X.

   (b) X is locally path-connected.

   (c) X is a normal space.

   Hint 1. Prove by induction on n, using the Tietze extension theorem, that for all closed subsets A and B of X, there exists continuous maps $f_n : X_n \to I$ for all $n \geq 0$ such that $f_n(A \cap X_n) = \{0\}$, $f_n(B \cap X_n) = \{1\}$ and $f_n|_{X_{n-1}} = f_{n-1}$.

2. Show that if X and Y are CW-complexes, and X and Y both have countably many cells, then $X \times Y$ is also a CW-complex.

   Hint 2. For all $0 \leq k \leq n$,

$$(D^n, S^{n-1}) \cong (D^k \times D^{n-k}, S^{k-1} \times D^{n-k} \cup D^k \times S^{n-k-1}).$$

3. Let X and Y be CW complexes, and let $f : X \to Y$ be a continuous map that is *cellular*, i.e., $f(X_n) \subset Y_n$ for all n. Prove that the mapping cylinder M_f is naturally a CW complex.

4. The *smash product* of two pointed spaces (X, x_0) and (Y, y_0) is the space

$$X \wedge Y = X \times Y / X \vee Y,$$

   with basepoint equal to the equivalence class of (x_0, y_0).

   (a) Show that $S^1 \wedge X \cong \Sigma X$, for all (X, x_0).

   (b) Let X and Y be CW-complexes such that $X_{r-1} = \{x_0\}$ and $Y_{s-1} = \{y_0\}$. Show that $X \wedge Y$ is a CW-complex with $(X \wedge Y)_{r+s-1} = \{[x_0, y_0]\}$.