Homotopie et Homologie
Exercise Set 1

19.09.2013

1. A morphism $f : A \to B$ in a category $\mathscr{C}$ is an *isomorphism* if there exists another morphism $g : B \to A$ in $\mathscr{C}$ such that $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$.

   (a) What are the isomorphisms in Set, Top, and Gr? What about in the category G arising from a group G?

   (b) Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Show that if $f : A \to B$ is an isomorphism in $\mathscr{C}$, then $F_1(f)$ is an isomorphism in $\mathscr{D}$. Does the converse hold?

2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Show that if we set

$$\mathrm{Ob}(\mathscr{C} \times \mathscr{D}) = \mathrm{Ob}\,\mathscr{C} \times \mathrm{Ob}\,\mathscr{D}$$

   and for all $C, C' \in \mathrm{Ob}\,\mathscr{C}$, $D, D' \in \mathrm{Ob}\,\mathscr{D}$

$$(\mathscr{C} \times \mathscr{D})\big((C, D), (C', D')\big) = \mathscr{C}(C, C') \times \mathscr{D}(D, D'),$$

   then there is a composition law making $\mathscr{C} \times \mathscr{D}$ into a category such that projection onto the first (respectively, second) component induces functors $P_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $P_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$. Show moreover that for every pair of functors $F : \mathscr{E} \to \mathscr{C}$ and $G : \mathscr{E} \to \mathscr{D}$, there is a unique functor $(F, G) : \mathscr{E} \to \mathscr{C} \times \mathscr{D}$ such that $P_1 \circ (F, G) = F$ and $P_2 \circ (F, G) = G$.

3. (Categories of functors)

   (a) Let $\mathscr{C}$ and $\mathscr{D}$ be small categories. Let $\mathscr{C}^{\mathscr{D}}$ denote the category, the objects of which are functors $F : \mathscr{D} \to \mathscr{C}$ and the morphisms of which are natural transformations between functors from $\mathscr{D}$ to $\mathscr{C}$. The objects of $\mathscr{C}^{\mathscr{D}}$ are often called *diagrams in $\mathscr{C}$ of shape $\mathscr{D}$*.

   (b) Show that a functor $G : \mathscr{B} \to \mathscr{C}$ induces a functor $G_* : \mathscr{B}^{\mathscr{D}} \to \mathscr{C}^{\mathscr{D}}$, while a functor $H : \mathscr{E} \to \mathscr{D}$ induces a functor $H^* : \mathscr{C}^{\mathscr{D}} \to \mathscr{C}^{\mathscr{E}}$.

   (c) Show that there is a large category Cat, the objects of which are small categories and the morphisms of which are functors between small categories.

4. Let $(X, \mathscr{T})$ be a topological space. Explain how to form a category $\Pi_1(X, \mathscr{T})$ with $\operatorname{Ob} \Pi_1(X, \mathscr{T}) = X$ and such that $\Pi_1(X, \mathscr{T})(x, x')$ is the set of all homotopy classes of paths from x to x'. Show that every morphism in $\Pi_1(X, \mathscr{T})$ is an isomorphism. (This category is called the *fundamental groupoid* of the space $(X, \mathscr{T})$.)

5. What is the pushout of the inclusion map $\iota : S^1 \hookrightarrow D^2$ with itself? What is the pullback of
$$p : \operatorname{Map}(I, Y) \to Y \times Y : \lambda \mapsto \big(\lambda(0), \lambda(1)\big)$$
with the diagonal map
$$\Delta : Y \to Y \times Y : y \mapsto (y, y)?$$
(Note that p is continuous, since I is locally compact and Hausdorff.)

6. Let X be a topological space. Recall the definition of the *cone* CX and the *suspension* ΣX on X from Series 23 of the topology course last year (http://wiki.epfl.ch/topologie1213/documents/Séries/Série23.pdf).

   Let $\iota_X : X \to CX$ denote the obvious inclusion of X into CX. Prove that the pushout of ι_X and the unique map $X \to \{*\}$ is homeomorphic to ΣX.