CELLULARIZATION OF CLASSIFYING SPACES AND FUSION PROPERTIES OF FINITE GROUPS

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ABSTRACT

One way to understand the mod p homotopy theory of classifying spaces of finite groups is to compute their $B\mathbb{Z}/p$-cellularization. In the easiest cases this is a classifying space of a finite group (always a finite p-group). If not, we show that it has infinitely many non-trivial homotopy groups. Moreover they are either p-torsion free or else infinitely many of them contain p-torsion. By means of techniques related to fusion systems we exhibit concrete examples where p-torsion appears and compute explicitly the cellularization.

Introduction

Let A be a pointed space. Although the idea of building a space as a homotopy colimit of copies of A goes back to Adams [1] in the framework of the classification of the acyclics of a certain generalized cohomology theory, it was in the early 1990s that Dror Farjoun [13] and Chachólski [11] formalized and developed this idea in the wider context of cellularity classes. Thus a space will be called *A-cellular* if it can be built from A by means of pointed homotopy colimits. There exists an A-cellularization functor CW_A that provides the best possible A-cellular approximation, in the sense that the natural map $\mathrm{CW}_A X \to X$ is an *A-equivalence*, that is, it induces a weak equivalence between pointed mapping spaces $\mathrm{Map}_*(A, \mathrm{CW}_A X) \simeq \mathrm{Map}_*(A, X)$.

Our interest lies essentially in using the cellularization functor to study the p-primary part of the homotopy of the classifying space of a finite group G. This approach was suggested by Dror Farjoun in [13, Example 3.C.9] and has proved to be very fruitful in the last years; we can mention work of Bousfield [5] which describes cellularization of nilpotent spaces with regard to Moore spaces $M(\mathbb{Z}/p, n)$ or the relationship recently discovered [28] between the $M(\mathbb{Z}/p, 1)$-cellularization of spaces and the $\mathbb{Z}/p$-cellularization in the category of groups.

In the present paper we focus our attention on describing the $B\mathbb{Z}/p$-cellularization of BG, where G is a finite group. From the above description of the cellularization functor, we know this is a space which can be built from $B\mathbb{Z}/p$ by means of push-outs, wedges, and telescopes, and which encodes all the information about the pointed mapping space $\mathrm{Map}_*(B\mathbb{Z}/p, BG)$, which is isomorphic to $\mathrm{Hom}(\mathbb{Z}/p, G)$. A first attempt to understand $\mathrm{CW}_{B\mathbb{Z}/p}BG$ was undertaken in [18], where in particular the fundamental group of the cellularization was described as an extension of the group-theoretical $\mathbb{Z}/p$-cellularization of a subgroup of G by a finite p-torsion free abelian group (see Section 1 for details). In the present note we concentrate thus on the higher homotopy groups $\pi_n(\mathrm{CW}_{B\mathbb{Z}/p}BG)$, for $n \geqslant 2$.

Our first result establishes that, in general, the space $\mathrm{CW}_{B\mathbb{Z}/p}BG$ has infinitely many non-trivial homotopy groups. This contrasts with the results in [10]: for easier spaces to work with, such as H-spaces or classifying spaces of nilpotent groups, the assumption that the mapping space $\mathrm{Map}_*(B\mathbb{Z}/p, X)$ is discrete implies that $\mathrm{CW}_{B\mathbb{Z}/p}X$ is aspherical.

PROPOSITION 2.3.   *Let G be a finite group. Then $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is either the classifying space of a finite p-group, or it has infinitely many non-trivial homotopy groups.*

An important role in the proof of this result is played by Levi's dichotomy theorem [**24**, Theorem 1.1.4] about the homotopy structure of the Bousfield–Kan p-completion of BG, which will be denoted by $BG_p^\wedge$ and simply called 'p-completion' throughout this paper [**6**]. We remark here that the homotopy groups of a $B\mathbb{Z}/p$-cellular space are not necessarily p-groups. We wish therefore to understand not only the p-primary part, but also the more accessible p'-primary part. The latter is indeed closely related to the $B\mathbb{Z}/p$-nullification of BG, due to Chachólski's description [**11**, Theorem 20.5] of the cellularization as a small variation of the homotopy fiber of the $B\mathbb{Z}/p$-nullification map (for information about the main properties of the nullification P_A and the cellularization CW_A, see [**11**, **13**]). The first author identified in [**11**, Theorem 3.5] the $B\mathbb{Z}/p$-nullification $P_{B\mathbb{Z}/p}BG$, when G is generated by elements of order p, as the product (taken over all primes q different from p) of the q-completions $BG_q^\wedge$.

THEOREM 2.5.   *Let G be a finite group generated by elements of order p. Then either the cellularization $\mathrm{CW}_{B\mathbb{Z}/p}BG$ has infinitely many homotopy groups containing p-torsion or it fits in a fibration*

$$\mathrm{CW}_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow \prod_{q \neq p} BG_q^\wedge,$$

where the (finite) product is taken over all primes q dividing the order of G, and the right map is the product of the completions.

This result can be used to compute the cellularization for an arbitrary finite group G. It suffices to replace the group by its subgroup $\Omega_1(G)$ generated by the elements of order p; see Section 1 for details.

Note also that in the second case the higher homotopy groups of the cellularization are those of $\Omega(\prod_{q \neq p} BG_q^\wedge)$. In particular, if G is generated by elements of order p, then $\mathrm{CW}_{B\mathbb{Z}/p}BG$ coincides with the homotopy fiber of the nullification map $\bar{P}_{B\mathbb{Z}/p}$ unless there appears (a lot of) p-torsion in the higher homotopy groups. The main question that remains unanswered up to this point is thus to determine if there actually exist groups for which $\mathrm{CW}_{B\mathbb{Z}/p}BG$ does contain p-torsion in its higher homotopy groups! This turns out to depend heavily on the p-complete classifying space $BG_p^\wedge$.

COROLLARY 3.3.   *Let G be a group generated by order p elements. Then the universal cover of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is p-torsion free if and only if the p-completion of BG is $B\mathbb{Z}/p$-cellular.*

We notice that the presence of p-torsion in the higher homotopy groups of the cellularization of BG depends on the fusion properties of G. If S is a Sylow p-subgroup, then define $\mathrm{Cl}(S)$ as the smallest *strongly closed* subgroup of S containing all order p elements of S. This means that whenever a conjugate gxg^{-1} of an element $x \in \mathrm{Cl}(S)$ by an element $g \in G$ lies in S, then in fact $gxg^{-1} \in \mathrm{Cl}(S)$. Our next result shifts the problem to the study of fusion systems.

THEOREM 4.2.   *Let G be a group generated by elements of order p. If S is equal to $\mathrm{Cl}(S)$, then the universal cover of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is p-torsion free.*

There are very few groups containing a proper strongly closed subgroup. This yields many examples of $B\mathbb{Z}/p$-cellular spaces with higher homotopy that is p-torsion free, including the cellularization of classifying spaces of the symmetric groups; see Example 4.8. It is a delicate

problem to obtain an explicit description of the p-primary part of $\mathrm{CW}_{B\mathbb{Z}/p}BG$. For this purpose we focus on groups in which the normalizer of a Sylow p-subgroup controls the fusion.

THEOREM 5.6.   *Let G be a finite group generated by elements of order p and S a Sylow p-subgroup. Assume that the normalizer $N_G(S)$ controls the fusion in G. Then $\mathrm{CW}_{B\mathbb{Z}/p}BG$ fits in a fibration*

$$\mathrm{CW}_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow B\Gamma_p^\wedge \times \prod_{q\neq p} BG_q^\wedge,$$

where $\Gamma = N_G(S)/\mathrm{Cl}(S)$.

At the prime 2, Foote actually characterized in [**19**] the groups containing a proper strongly closed subgroup. In Corollary 5.8, we apply the previous theorem to obtain a description of the only two families of simple groups for which 2-torsion appears in the higher homotopy groups of $\mathrm{CW}_{B\mathbb{Z}/2}BG$. For example, there is a fibration

$$\mathrm{CW}_{B\mathbb{Z}/2}B\mathrm{Sz}(2^n) \to B\mathrm{Sz}(2^n) \to B\big((\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1)\big)_2^\wedge \times \prod_{q\neq 2} B\mathrm{Sz}(2^n)_q^\wedge$$

and a similar description holds for the groups $U_3(2^n)$. At odd primes we offer another concrete calculation in Corollary 5.9 for certain projective special linear groups, but we do not claim that they are the unique simple groups for which p-torsion appears. In the last section we find explicit 'exotic' representations (which are not induced by a group homomorphism) of these groups in compact Lie groups.

1. *Background*

There are two main ingredients used to analyse the cellularization $\mathrm{CW}_{B\mathbb{Z}/p}BG$. The first one is a general recipe, the second a specific simplification in the setting of classifying spaces of finite groups.

One of the most efficient tools to compute cellularization functors is Chachólski's construction of CW_A out of the nullification functor P_A, see [**11**]. His main theorem states that $\mathrm{CW}_A X$ can be constructed in two steps. First consider the evaluation map $\bigvee_{[A,X]_*} A \to X$ (the wedge is taken over representatives of all pointed homotopy classes of maps) and let C denote its homotopy cofiber. Then $\mathrm{CW}_A X$ is the homotopy fiber of the composite map $X \to C \to P_{\Sigma A}C$. We will actually use a small variation of Chachólski's description of the cellularization, and this is the construction we will refer to in the text.

THEOREM 1.1.   *Let A and X be pointed connected spaces and choose a pointed map $A \to X$ representing each unpointed homotopy class in $[A, X]$. Denote by C the homotopy cofiber of the evaluation map $\mathrm{ev} : \bigvee_{[A,X]} A \to X$. Then $\mathrm{CW}_A X$ is weakly equivalent to the homotopy fiber of the composite $X \to C \to P_{\Sigma A}C$.*

Proof. According to [**11**, Theorem 20.3] one has to check that composing any map $f : A \to X$ with $\mathrm{ev} : X \to C$ is null-homotopic. However, such a map f is freely homotopic to one in $[A, X]$. Thus $\mathrm{ev} \circ f$ is freely homotopic to the constant map, and so it must also be null-homotopic in the pointed category. $\square$

In [**18**], the first author focused on the situation when $A = B\mathbb{Z}/p$ and X is the classifying space of a finite group G. A first reduction can always be done by considering the subgroup $\Omega_1(G)_p$ of G generated by the elements of order p. This notation, or the shorter notation $\Omega_1(G)$ when the prime p is understood, is the standard one in group theory; see for example [**21**]. In [**18**, **28**], the terminology 'socle' and the corresponding notation $S_{\mathbb{Z}/p}G$ was used instead. We have an equivalence of pointed mapping spaces $\mathrm{Map}_*(B\mathbb{Z}/p, BG) \simeq \mathrm{Map}_*(B\mathbb{Z}/p, B\Omega_1(G))$, and this means that $B\Omega_1(G) \to BG$ is a $B\mathbb{Z}/p$-cellular equivalence. This is why we may always assume that G is generated by elements of order p.

A second simplification consists then in computing the group theoretical cellularization $\mathrm{CW}_{\mathbb{Z}/p}G$ of G, which was developed in [**28**] as an analog in the category of groups of the aforementioned cellularization of spaces. The group $\mathrm{CW}_{\mathbb{Z}/p}G \cong \mathrm{CW}_{\mathbb{Z}/p}\Omega_1(G)$ is a (finite) central extension of $\Omega_1(G)$ by a group of order coprime with p. Since $B\mathrm{CW}_{\mathbb{Z}/p}G \to BG$ is a $B\mathbb{Z}/p$-cellular equivalence, one could assume that G is a finite $\mathbb{Z}/p$-cellular group (it can be constructed out of the cyclic group $\mathbb{Z}/p$ by iterated colimits). Examples of such groups are provided at the prime 2 by dihedral groups, Coxeter groups, and in general by finite p-groups generated by order p elements. However, it is not so easy in practice to verify whether or not a group is $\mathbb{Z}/p$-cellular. Therefore we prefer to state results about groups generated by elements of order p.

The first author described the fundamental group of $\mathrm{CW}_{B\mathbb{Z}/p}BG$.

PROPOSITION 1.2 [**18**, Theorem 4.14]. *Let G be a finite $\mathbb{Z}/p$-cellular group. Then the fundamental group $\pi = \pi_1\mathrm{CW}_{B\mathbb{Z}/p}BG$ is described as an extension $H \hookrightarrow \pi \twoheadrightarrow G$ of G by a finite p-torsion free abelian group H.*

He also showed that BG is $B\mathbb{Z}/p$-cellular if and only if G is a finite $\mathbb{Z}/p$-cellular p-group. However, in general, the cellularization $\mathrm{CW}_{B\mathbb{Z}/p}BG$ differs vastly from BG, even when G is $\mathbb{Z}/p$-cellular, as illustrated by Example 2.6. This paper is an attempt to understand $\mathrm{CW}_{B\mathbb{Z}/p}BG$ when G is not a p-group.

2. *A cellular dichotomy*

We establish in this section that the space $\mathrm{CW}_{B\mathbb{Z}/p}BG$ has infinitely many non-trivial homotopy groups unless it is aspherical. We will say more about the higher homotopy groups in the next section. Recall that C is the Chachólski cofiber introduced in Theorem 1.1.

LEMMA 2.1. *Let G be a finite group generated by elements of order p. Then the simply connected space $P_{\Sigma B\mathbb{Z}/p}C$ is homotopy equivalent to $\prod_q BG_q^{\wedge} \times (P_{\Sigma B\mathbb{Z}/p}C)_p^{\wedge}$, where the product is taken over all primes $q \neq p$.*

Proof. Since G is generated by order p elements, the evaluation map $*\mathbb{Z}/p \to G$ taken over all homomorphisms $\mathbb{Z}/p \to G$ is surjective. Hence the homotopy cofiber C of the evaluation map $\bigvee_{[B\mathbb{Z}/p, BG]} B\mathbb{Z}/p \to BG$ is a simply connected space. Recall from [**4**, 2.8] that the nullification $P_{\Sigma B\mathbb{Z}/p}C$ is constructed from C by taking iterated homotopy cofibers of maps from $\Sigma^k B\mathbb{Z}/p$, with $k \geqslant 1$. In particular, $P_{\Sigma B\mathbb{Z}/p}C$ is simply connected as well.

We want to analyse the q-completion of this last space for a prime $q \neq p$. However, since $B\mathbb{Z}/p$ and its suspensions are $H\mathbb{Z}/q$-acyclic, we see that the composite

$$BG \longrightarrow C \longrightarrow P_{\Sigma B\mathbb{Z}/p}C$$

is an equivalence in mod q homology, which implies that $(P_{\Sigma B\mathbb{Z}/p}C)_q^\wedge \simeq BG_q^\wedge$. Finally, as $B\mathbb{Z}/p$ and BG are rationally trivial, we use Sullivan's arithmetic square [**6**, Lemma VI.8.1] to establish that $P_{\Sigma B\mathbb{Z}/p}C$ splits as product of its q-completions, taken over all primes q. $\qquad\square$

REMARK 2.2. In the proof of the previous lemma we used the fact that if A is an acyclic space for a certain homology theory h_*, then the nullification map $X \to P_A X$ is an h_*-equivalence for any space X. This is an easy consequence of the construction of $P_A X$ by adjoining cones on maps $\Sigma^k A \to X$ and the associated Mayer–Vietoris sequences.

In the next proposition we will use the fact that a group generated by elements of order p is q-perfect for any prime $q \neq p$. A finite group G is called q-perfect if G has no proper quotients of order a power of q, or equivalently if $H_1(G; \mathbb{F}_q) = 0$.

PROPOSITION 2.3. Let G be a finite group. Then $CW_{B\mathbb{Z}/p}BG$ is either the classifying space of a finite p-group, or it has infinitely many non-trivial homotopy groups.

Proof. To understand $CW_{B\mathbb{Z}/p}BG$, we have seen in the previous section that we can always replace G by the subgroup $\Omega_1(G)$. Thus, we can assume that G is generated by elements of order p. Now, if $CW_{B\mathbb{Z}/p}BG$ is the classifying space of a group, then G must be a finite p-group as shown by the first author in [**18**, Theorem 4.14]. Assume therefore that it is not so, that is, there exists a prime q different from p dividing the order of G.

By Levi's result [**24**, Theorem 1.1.4] for q-completions of classifying spaces of finite groups one infers that $BG_q^\wedge$ must have infinitely many non-trivial homotopy groups; this is because G is q-perfect as its abelianization must be p-torsion. Consider now the long exact sequence in homotopy of the fibration from Theorem 1.1 $CW_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow P_{\Sigma B\mathbb{Z}/p}C$ and conclude by using Lemma 2.1. $\qquad\square$

In view of the proof of the preceding theorem, the main question is to determine whether or not the higher homotopy groups of the cellularization of classifying spaces may contain p-torsion. We make next an observation about $\Sigma B\mathbb{Z}/p$-null spaces which play such an important role for the $B\mathbb{Z}/p$-cellularization because of Chachólski's Theorem 1.1. The proof is very much in the spirit of the original theorem of Serre [**29**, Théorème 10], the Dwyer–Wilkerson result [**16**, Theorem 1.3], or Levi's one [**24**, Theorem 1.1.4]; compare also Grodal's work on Postnikov pieces [**22**].

PROPOSITION 2.4. Let X be a simply connected torsion $\Sigma B\mathbb{Z}/p$-null space. Then it is either p-torsion free or has infinitely many homotopy groups containing p-torsion.

Proof. Let n be an integer greater than 1 and consider the Postnikov fibration

$$X\langle n\rangle \longrightarrow X\langle n-1\rangle \longrightarrow K(\pi_n X, n),$$

where $X\langle n\rangle$ stands for the n-connected cover of X. Since the base point component of the iterated loop space $\Omega^{n-1}X$ and $\Omega^{n-1}(X\langle n-1\rangle)$ are weakly equivalent there is a fibration

$$\Omega_0^{n-1}X \longrightarrow K(\pi_n X, 1) \longrightarrow \Omega^{n-2}(X\langle n\rangle)$$

in which the fiber is $B\mathbb{Z}/p$-null. Thus the total space is $B\mathbb{Z}/p$-null if and only if the base is so, that is, the homotopy group $\pi_n X$ contains p-torsion if and only if the n-connected cover

$X\langle n\rangle$ has some p-torsion. We use here that a simply connected p'-torsion space Y has trivial p-completion, and therefore the pointed mapping space $\operatorname{Map}_*(B\mathbb{Z}/p, Y)$ is contractible. $\qquad\square$

We turn now to a more detailed study of the case when G is not a p-group. We have seen that $\operatorname{CW}_{B\mathbb{Z}/p}BG$ has infinitely many non-trivial homotopy groups, but they can arise in two different ways, because the space $P_{\Sigma B\mathbb{Z}/p}C$ may contain p-torsion.

THEOREM 2.5. *Let G be a finite group generated by order p elements. Then either the cellularization $\operatorname{CW}_{B\mathbb{Z}/p}BG$ has infinitely many homotopy groups containing p-torsion or it fits in a fibration*

$$\operatorname{CW}_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow \prod_{q\neq p} BG_q^\wedge,$$

where the (finite) product is taken over all primes q dividing the order of G, and the right map is the product of the completions.

Proof. Because the group G is finite, its reduced integral homology groups are finite. The space $P_{\Sigma B\mathbb{Z}/p}C$ is constructed out of BG by taking iterated homotopy cofibers of maps out of (finite) wedges of suspensions of $B\mathbb{Z}/p$, with homology groups that are all torsion, and thus $P_{\Sigma B\mathbb{Z}/p}C$ satisfies the conditions of the above proposition.

Assume thus that $P_{\Sigma B\mathbb{Z}/p}C$ is p-torsion free. We see then by Lemma 2.1 that $P_{\Sigma B\mathbb{Z}/p}C$ is weakly equivalent to the product $\prod_{q\neq p} BG_q^\wedge$. $\qquad\square$

We remark that the computation in [**18**, Section 3] allows to identify the cellularization of BG with $\bar{P}_{B\mathbb{Z}/p}BG$, the homotopy fiber of the nullification map, when the space $P_{\Sigma B\mathbb{Z}/p}C$ is p-torsion free.

EXAMPLE 2.6. Consider the C_2-cellular group Σ_3, symmetric group on three letters. The choice of any transposition yields a map $f : B\mathbb{Z}/2 \to B\Sigma_3$ which induces an isomorphism in mod 2 homology. Therefore the homotopy cofiber C_f of f is 2-torsion free. The proof of Theorem 2.5 shows that $\operatorname{CW}_{B\mathbb{Z}/2}B\Sigma_3$ is a space with fundamental group Σ_3 and its universal cover is $\Omega(B\Sigma_3)_3^\wedge \simeq S^3\{3\}$, the homotopy fiber of the degree 3 map on the sphere S^3; compare [**6**, Proposition VII.4.4; **28**, Theorem 7.5].

3. *The p-completed classifying space*

The example of the symmetric group Σ_3 might lead us to think that there will be very few cases when the universal cover of $\operatorname{CW}_{B\mathbb{Z}/p}BG$ is p-torsion free. In Section 4 we will see that there are surprisingly many groups for which this occurs. Let us first see how the properties of the cellularization we are looking for are reflected in the classifying space completed at p.

We start with a general result telling how to detect the presence of p-torsion in the cellularization of BG.

PROPOSITION 3.1. *Let G be a finite group generated by elements of order p. For $\operatorname{CW}_{B\mathbb{Z}/p}BG$ to have infinitely many homotopy groups with p-torsion, there must exist a p-complete space Z which is $\Sigma B\mathbb{Z}/p$-null and a map $f : BG \to Z$ which is not null-homotopic such that the restriction $B\mathbb{Z}/p \to BG \to Z$ to any cyclic subgroup of order p is null-homotopic.*

Proof. Consider again the cofibration $\vee B\mathbb{Z}/p \to BG \to C$. The cofiber C is a simply connected torsion space, and hence equivalent to the finite product $\prod C_q^\wedge$, where q runs over the primes dividing the order of G. We need to understand the $\Sigma B\mathbb{Z}/p$-nullification of C. Since

nullification commutes with finite products we only look at $C_p^\wedge$ and infer that $\mathrm{CW}_{B\mathbb{Z}/p}BG$ has infinitely many homotopy groups with p-torsion if and only if $P_{\Sigma B\mathbb{Z}/p}(C_p^\wedge)$ does so, which happens if and only if $P_{\Sigma B\mathbb{Z}/p}(C_p^\wedge)$ is not contractible by Theorem 2.5.

In other words we wish to know when the map $C_p^\wedge \to *$ is not a $\Sigma B\mathbb{Z}/p$-equivalence. This means by definition that there exists some $\Sigma B\mathbb{Z}/p$-null space Z for which the pointed mapping space $\mathrm{Map}_*(C_p^\wedge, Z)$ is not contractible. Because C is a simply connected homotopy cofiber of torsion spaces, $C_p^\wedge$ is a p-torsion space. Thus we can assume, using Sullivan's arithmetic square, that Z is p-complete.

The cofibration sequence $\vee B\mathbb{Z}/p \to BG \to C$ yields a fibration

$$\mathrm{Map}_*(C, Z) \to \mathrm{Map}_*(BG, Z) \to \prod \mathrm{Map}_*(B\mathbb{Z}/p, Z)$$

in which the loop space of the base is trivial since Z is $\Sigma B\mathbb{Z}/p$-null. The existence of the map f tells us that there is a component of the total space (different from the component of the constant map) which lies over the component of the base point in the base. Therefore $\mathrm{Map}_*(C, Z)$ has at least two components as well. As Z is p-complete, $\mathrm{Map}_*(C, Z)$ is weakly equivalent to $\mathrm{Map}_*(C_p^\wedge, Z)$.

On the other hand, if no such map f exists, then $\mathrm{Map}_*(C, Z)$ is weakly equivalent to the connected component of the constant $\mathrm{Map}_*(BG, Z)_c$, which is contractible by Dwyer's result [**14**, Theorem 1.2]. $\qquad\square$

We compare now the cellularization of the classifying space BG and its p-completion. Recall from [**6**, II.5], see also [**24**], that the fundamental group of $BG_p^\wedge$ is always isomorphic to the group theoretical p-completion, that is, the quotient of G by $O^p(G)$, the maximal p-perfect subgroup of G.

PROPOSITION 3.2. *Let G be a finite group generated by elements of order p. Then $(\mathrm{CW}_{B\mathbb{Z}/p}BG)_p^\wedge$ is homotopy equivalent to $\mathrm{CW}_{B\mathbb{Z}/p}(BG_p^\wedge)$.*

Proof. We know for example from [**17**] that the completion map $BG \longrightarrow BG_p^\wedge$ induces a chain of bijections identifying the set of unpointed homotopy classes

$$[B\mathbb{Z}/p, BG] \simeq \left[B\mathbb{Z}/p, BG_p^\wedge\right] \simeq \mathrm{Rep}\left(\mathbb{Z}/p, G\right).$$

Choose a set of representatives $f : B\mathbb{Z}/p \to BG$ for all conjugacy classes of elements of order p in G and write $f_p^\wedge$ for the corresponding map into the p-completion of BG. Consider this diagram of cofibrations.

$$
\begin{array}{ccccc}
\bigvee B\mathbb{Z}/p & \xrightarrow{\vee f} & BG & \longrightarrow & C \\
\| & & \downarrow & & \downarrow \\
\bigvee B\mathbb{Z}/p & \xrightarrow{\vee f_p^\wedge} & BG_p^\wedge & \longrightarrow & D
\end{array}
$$

We know from Lemma 2.1 that $P_{\Sigma B\mathbb{Z}/p}C \simeq P_{\Sigma B\mathbb{Z}/p}C_p^\wedge \times \prod_{q \neq p} BG_q^\wedge$. Comparing next the Mayer–Vietoris sequences in mod p homology of the previous two cofibrations, we see that $C_p^\wedge \simeq D$.

The arithmetic square argument in Lemma 2.1, applied this time to D, yields an equivalence $P_{\Sigma B\mathbb{Z}/p}D \simeq (P_{\Sigma B\mathbb{Z}/p}D)_p^\wedge$, and thus $P_{\Sigma B\mathbb{Z}/p}D$ is p-complete. Finally, since the base spaces $P_{\Sigma B\mathbb{Z}/p}C$ and $P_{\Sigma B\mathbb{Z}/p}D$ in Chachólski's fibrations are simply connected, the p-completion of these fibrations are again fibrations by the nilpotent fiber lemma [**6**, Lemma II.5.1]. Therefore $\mathrm{CW}_{B\mathbb{Z}/p}(BG_p^\wedge)$ is p-complete and coincides with $(\mathrm{CW}_{B\mathbb{Z}/p}BG)_p^\wedge$. $\qquad\square$

As a consequence, the existence of p-torsion in the upper homotopy of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is strongly related with the $B\mathbb{Z}/p$-cellularity of $BG_p^\wedge$.

COROLLARY 3.3. *Let G be a finite group generated by order p elements. Then the universal cover of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is p-torsion free if and only if the p-completion of BG is $B\mathbb{Z}/p$-cellular.*

Proof. By the previous proposition the p-completion of BG is $B\mathbb{Z}/p$-cellular if and only if the p-completion of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is so. From the fibration

$$\mathrm{CW}_{B\mathbb{Z}/p}(BG_p^\wedge) \to BG_p^\wedge \to P_{\Sigma B\mathbb{Z}/p}D,$$

which is the p-completion of the corresponding fibration for BG, we see that this happens precisely when the space D is $\Sigma B\mathbb{Z}/p$-acyclic, that is, when the space $P_{\Sigma B\mathbb{Z}/p}C$ is p-torsion free. $\qquad\square$

4. Strongly closed subgroups

We recall now (see for example [**19**]) the notion of strongly closed subgroup. The main result of the section is that the existence of a proper strongly closed subgroup in a Sylow p-subgroup is an unavoidable condition to find p-torsion in the $B\mathbb{Z}/p$-cellularization of BG.

DEFINITION 4.1. Let G be a finite group and H a subgroup of some Sylow p-subgroup S of G. Then H is *strongly closed* in G if whenever $h \in H$ and $g \in G$ are such that $ghg^{-1} \in S$, then $ghg^{-1} \in H$.

We consider next the smallest strongly closed subgroup $\mathrm{Cl}(S)$ of S containing $\Omega_1(S)$. It can be built inductively in a finite number of steps starting from $\Omega_1(S) = \mathrm{Cl}_0(S)$, and constructing $\mathrm{Cl}_{i+1}(S)$ from $\mathrm{Cl}_i(S)$ by adding all conjugates of elements of $\mathrm{Cl}_i(S)$ by elements of G which belong to S, that is, $\mathrm{Cl}_{i+1}(S)$ is generated by all elements $gxg^{-1} \in S$ with $g \in G$ and $x \in \mathrm{Cl}_i(S)$. Then $\mathrm{Cl}(S) = \bigcup_i \mathrm{Cl}_i(S)$.

We start with an observation which leads then naturally to a closer analysis of the fusion of the groups we look at.

THEOREM 4.2. *Let G be a group generated by elements of order p. If S is equal to $\mathrm{Cl}(S)$, then the universal cover of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is p-torsion free.*

Proof. By Proposition 3.1, the existence of p-torsion is detected by a non-trivial map $f : BG_p^\wedge \to Z$ into some $\Sigma B\mathbb{Z}/p$-null space Z, which is trivial when restricted to any cyclic subgroup $\mathbb{Z}/p$ in G. The composite $BG \to Z \to K(\pi_1 Z, 1)$ is then null-homotopic because it corresponds to a homomorphism $G \to \pi_1 Z$ restricting trivially to all generators of G. Therefore the map f lifts to the universal cover of Z and so we might assume that Z is 1-connected. We can also p-complete it if necessary, so that Z is actually $H\mathbb{Z}/p$-local.

By Dwyer's theorem [**14**, Theorem 1.4], the map f is null-homotopic if and only if the restriction $\bar{f}$ to some Sylow p-subgroup S of G is so. As we assume that $S = \mathrm{Cl}(S)$ we will show by induction that the restriction of f to $\mathrm{Cl}_i(S)$ is null-homotopic for all i. Clearly the restriction to $\mathrm{Cl}_0(S) = \Omega_1(S)$ is so, because this subgroup is generated by elements of order p. Assume now that the restriction to $\mathrm{Cl}_i(S)$ is null-homotopic, and consider the extension $\mathrm{Cl}_i(S) < \mathrm{Cl}_{i+1}(S) \twoheadrightarrow \mathrm{Cl}_{i+1}(S)/\mathrm{Cl}_i(S)$. By induction hypothesis the composite

$$B\mathrm{Cl}_i(S) \to BS \xrightarrow{\bar{f}} Z$$

is null-homotopic. Hence the map $\bar{f}$ restricted to $B\mathrm{Cl}_{i+1}(S)$ factors through a map $h : B\big(\mathrm{Cl}_{i+1}(S)/\mathrm{Cl}_i(S)\big) \to Z$ by Zabrodsky's lemma [**14**, Proposition 3.5] and it is null-homotopic

if and only if h is so. Since generators of the quotient group are classes of conjugates of elements in $\mathrm{Cl}_i(S)$, on which $\bar{f}$ restricts trivially, we see that h is null-homotopic when restricted to a set of generators. Therefore h is null-homotopic (cf. [**10**, Proposition 2.4] for a more detailed account), and so must be $\bar{f}$. $\square$

In many cases the strongly closed subgroup $\mathrm{Cl}(S)$ is equal to S for obvious reasons, for example when $S = \Omega_1(S)$.

COROLLARY 4.3. *Let G be a group generated by elements of order p with Sylow p-subgroup that is generated by elements of order p as well. The universal cover of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is then p-torsion free.*

EXAMPLE 4.4. Consider the C_2-cellular group Σ_{2^n}, symmetric group on 2^n letters with $n \geqslant 2$. The Sylow 2-subgroup is an iterated wreath product of copies of $\mathbb{Z}/2$ which is always generated by elements of order 2. Therefore the above proposition applies and we obtain that $P_{\Sigma B\mathbb{Z}/2}C \simeq \prod_{q\neq 2}(B\Sigma_{2^n})_q^{\wedge}$. Thus $\mathrm{CW}_{B\mathbb{Z}/2}B\Sigma_{2^n}$ fits into a fibration

$$\prod_{q\neq 2}\Omega\big((B\Sigma_{2^n})_q^{\wedge}\big) \to \mathrm{CW}_{B\mathbb{Z}/2}B\Sigma_{2^n} \to B\Sigma_{2^n}.$$

Our second corollary deals with the case when the Sylow p-subgroup is possibly not generated by elements of order p, but is nevertheless equal to $\mathrm{Cl}_1(S)$.

COROLLARY 4.5. *Let G be a a group generated by elements of order p with Sylow p-subgroup that is generated by $\Omega_1(S)$ and all their conjugates (by elements in G) which belong to S. The universal cover of $\mathrm{CW}_{B\mathbb{Z}/p}BG$ is then p-torsion free.*

EXAMPLE 4.6. The symmetric group Σ_3 acts by permutation on $(\mathbb{Z}/4)^3$. The diagonal is invariant, and so is the 'orthogonal' subgroup $\mathbb{Z}/4 \times \mathbb{Z}/4$. We define G to be the semi-direct product of $\mathbb{Z}/4 \times \mathbb{Z}/4$ by Σ_3. It is easy to check that G is generated by elements of order 2, but the Sylow 2-subgroup $S = (\mathbb{Z}/4 \times \mathbb{Z}/4) \rtimes \mathbb{Z}/2$ is not. The subgroup $\Omega_1(S)$ has index 2, and a representative of the generator of the quotient can be taken inside S to have order four (it is inside of G a product of three elements of order 2). This element has a conjugate which lies inside $\Omega_1(S)$, so we may conclude by the above corollary that $\mathrm{CW}_{B\mathbb{Z}/2}BG$ is 2-torsion free, that is, its universal cover is $\Omega(BG_3^{\wedge})$ by Theorem 2.5.

EXAMPLE 4.7. The group $G = \mathrm{PSL}_3(3)$ presents the same features as the above example. It is generated by elements of order 2 (it is simple), but its Sylow 2-subgroup S is semi-dihedral of order 16. The subgroup $\Omega_1(S)$ is dihedral of order 8. Here as well the universal cover of $\mathrm{CW}_{B\mathbb{Z}/2}B\mathrm{PSL}_3(3)$ is $\Omega\big(B\mathrm{PSL}_3(3)_3^{\wedge}\big)$.

Let us finish this section by showing with some examples the applicability of the last result.

EXAMPLE 4.8. Consider the 2-completion of the classifying space of the symmetric group Σ_{2^n}. According to Example 4.4, the Sylow 2-subgroup of Σ_{2^n} is generated by order 2 elements, and moreover $\pi_k(\mathrm{CW}_{B\mathbb{Z}/2}B\Sigma_{2^n})$ is 2-torsion free if $k \geqslant 2$. Thus by Proposition 3.3 $(B\Sigma_{2^n})_2^{\wedge}$ is $B\mathbb{Z}/2$-cellular.

Unlike what happens with the cellularization of BG, our study does not give in general information about $\mathrm{CW}_{B\mathbb{Z}/p}(BG_p^{\wedge})$ when G is not generated by order p elements. In this case,

however, it is sometimes possible to reduce the problem to the case of groups for which the hypothesis of the theorem holds.

EXAMPLE 4.9. Let us compute the $B\mathbb{Z}/2$-cellularization of $(BA_4)_2^\wedge$. It is known that the natural inclusion $A_4 < A_5$ induces an equivalence in $\mathrm{mod}\, 2$ homology, and then $(BA_4)_2^\wedge \simeq (BA_5)_2^\wedge$. Now, as A_5 is simple with 2-torsion, it is generated by order 2 elements, and moreover the Sylow 2-subgroup of A_5 is the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Hence, $(BA_5)_2^\wedge$ is $B\mathbb{Z}/2$-cellular by Corollary 3.3, and then $(BA_4)_2^\wedge$ is so as well.

The computations we were able to perform up to now depend on the absence of p-torsion in the cellularization. In the following section we start a more systematic study and include explicit examples where p-torsion appears.

5.   Control of fusion

To find an example where $\mathrm{CW}_{B\mathbb{Z}/p}BG$ has p-torsion, one should look by Theorem 4.2 for groups generated by elements of order p with a Sylow p-subgroup which is not generated by elements of order p. Moreover the smallest strongly closed subgroup containing $\Omega_1(S)$ should be a proper subgroup of S. A possible source of examples is given by groups, where the subgroup $\Omega_1(S)$ has large index in S, and is preserved by fusion.

DEFINITION 5.1.   Let G be a finite group and S a Sylow p-subgroup. The normalizer $N_G(S)$ of the Sylow p-subgroup *controls fusion* if, whenever $P < G$ is a p-subgroup and $gPg^{-1} < N_G(S)$, we have $g = hc$, with $h \in N_G(S)$ and $c \in C_G(P)$. Moreover, a p-group P is called a *Swan group* if it is such that $N_G(P)$ controls p-fusion for each group G containing P as a Sylow p-subgroup.

So far several families of Swan groups have been found, as for example abelian p-groups [31], most of the generalized extraspecial p-groups [20, 30] or metacyclic p-groups for p odd [12]. In fact, almost all p-groups are Swan groups [25].

It is also well known that the inclusion $N_G(S) \hookrightarrow G$ induces an isomorphism in $\mathrm{mod}\, p$ cohomology if $N_G(S)$ controls fusion in G; see for example [25, Proposition 2.1]. In other words there is an equivalence $BN_G(S)_p^\wedge \simeq BG_p^\wedge$ of p-completed classifying spaces.

Next we describe some examples of these kinds of groups, which will be central in our study of the $B\mathbb{Z}/p$-cellularization.

EXAMPLE 5.2.   Consider the Suzuki group $\mathrm{Sz}(2^n)$, with n an odd integer at least 3. The section [21, Section 16.4] is extremely useful to understand its basic subgroup and fusion properties. In particular, the Sylow 2-subgroup of $\mathrm{Sz}(2^n)$ can be written as an extension $(\mathbb{Z}/2)^n \hookrightarrow S \twoheadrightarrow (\mathbb{Z}/2)^n$, where the kernel is the center of the group and contains all its order 2 elements. Observe in particular that $\Omega_1(S)$ is not equal to S and its index is 2^n. The normalizer of the Sylow 2-subgroup of $\mathrm{Sz}(2^n)$ is a semi-direct product $S \rtimes \mathbb{Z}/(2^n - 1)$ which is maximal in $\mathrm{Sz}(2^n)$.

There are various ways to see why this normalizer controls fusion in $\mathrm{Sz}(2^n)$. One can for example use Dwyer's normalizer decomposition [15, Proposition 7.17] for the collection of 2-centric and 2-radical subgroups (which is very much in the spirit of [9]).

The group $\mathrm{Sz}(2^n)$ has the property that all of its Sylow 2-subgroups are disjoint (they have trivial intersection), and hence the unique 2-centric, 2-radical subgroup (up to conjugation) is S itself. The outer automorphism group of S in the fusion system of the Suzuki group, that is, the quotient of the normalizer $N_{\mathrm{Sz}(2^n)}(S)$ by S, is cyclic of order $2^n - 1$, generated by an

element ϕ which acts fixed-point free and permutes transitively the non-trivial elements of the center of S (see [**21**, Section 16.4]).

Now it is clear that the inclusion $S \rtimes \mathbb{Z}/(2^n - 1) \hookrightarrow \mathrm{Sz}(2^n)$ induces a homotopy equivalence $f : B(S \rtimes \mathbb{Z}/(2^n - 1))_2^{\wedge} \simeq B\mathrm{Sz}(2^n)_2^{\wedge}$.

An alternative way is to use the Cartan–Eilenberg double coset formula [**2**, Theorem II.6.6]. From the fact that the Sylow 2-subgroups are disjoint it is easy to deduce that both inclusions $S \hookrightarrow \mathrm{Sz}(2^n)$ and $S \hookrightarrow N_{\mathrm{Sz}(2^n)}(S)$ induce monomorphisms in mod 2 cohomology with isomorphic images (of stable elements).

EXAMPLE 5.3.   Consider the simple groups $U_3(2^n)$, with $n \geqslant 2$. We again refer the reader to [**21**, Section 16.4] for the following facts. Any Sylow 2-subgroup S of $U_3(2^n)$ is a 2-group of order 2^{3n}. Here as well $Z(S) = \Omega_1(S) = \Phi(S)$ is elementary abelian of order 2^n, and it can be seen, as in the previous case, that the unique 2-central, 2-radical subgroup is the 2-Sylow. The normalizer of S is a semi-direct product $S \rtimes \mathbb{Z}/(2^{2n} - 1)d^{-1}$, where $d = 1$ if n is even, and $d = 3$ if n is odd. The normalizer $N_{U_3(2^n)}(S)$ is a maximal subgroup in $U_3(2^n)$.

At odd primes, the special linear groups give other examples.

EXAMPLE 5.4.   Let p be an odd prime, n an integer at least 2, and $q = mp^n + 1$. Consider the linear group $\mathrm{PSL}_2(q)$. According to [**21**, Lemma 15.1.1], the p-Sylow subgroup of $\mathrm{PSL}_2(q)$ is cyclic of order p^n, and moreover its normalizer $N_{\mathrm{PSL}_2(q)}(S)$ is isomorphic to the semi-direct product $\mathbb{Z}/p^n \rtimes \mathbb{Z}/2$, where the action is given by the change of sign. As the Sylow subgroup is abelian, it is a Swan group, and hence the inclusion $N_{\mathrm{PSL}_2(q)}(S) \hookrightarrow \mathrm{PSL}_2(q)$ induces a homotopy equivalence between the p-completions of the classifying spaces.

As in Section 4 let G be a finite group generated by elements of order p, S a Sylow p-subgroup, and $\mathrm{Cl}(S)$ the smallest strongly closed subgroup containing $\Omega_1(S)$. Assume that $N_G(S)$ controls fusion in G. Since every automorphism of S induced by conjugation in G restricts to an automorphism of $\mathrm{Cl}(S)$, we see that $\mathrm{Cl}(S)$ is normal in $N_G(S)$. In the sequel we will denote by Γ the quotient $N_G(S)/\mathrm{Cl}(S)$. Notice that Γ is p-perfect since an epimorphism $\Gamma \twoheadrightarrow \mathbb{Z}/p$ would yield a non-trivial map $BN_G(S)_p^{\wedge} \to B\mathbb{Z}/p$ which is null-homotopic when restricted to $B\mathrm{Cl}(S)$. However, since $N_G(S)$ controls fusion in G, this corresponds to an epimorphism $G \twoheadrightarrow \mathbb{Z}/p$ which is trivial on all elements of order p. Such a morphism cannot exist since G is generated by its order p elements. In the following proposition D is, as in Proposition 3.2, the cofiber of Chachólski's cofibration for $BG_p^{\wedge}$.

PROPOSITION 5.5.   *Let G be a finite group generated by elements of order p and assume that the normalizer $N_G(S)$ of a Sylow p-subgroup controls fusion in G. Then $P_{\Sigma B\mathbb{Z}/p}D$ has the homotopy type of $B\Gamma_p^{\wedge}$.*

Proof.   Consider the composition $BG_p^{\wedge} \xrightarrow{\simeq} BN_G(S)_p^{\wedge} \to B\Gamma_p^{\wedge}$ induced by the projection $N_G(S) \twoheadrightarrow \Gamma$. As $\Omega_1(S) \leqslant \mathrm{Cl}(S)$, this map factors through the cofiber D, and hence through $P_{\Sigma B\mathbb{Z}/p}D$, because $B\Gamma_p^{\wedge}$ is $\Sigma B\mathbb{Z}/p$-null. Thus, we have the following diagram.

$$
\begin{array}{ccccc}
BG_p^{\wedge} & \xrightarrow{\ h\ } & D & \xrightarrow{\ l\ } & P_{\Sigma B\mathbb{Z}/p}D \\
{\scriptstyle B\pi_p^{\wedge}}\Big\downarrow & {\scriptstyle g'}\swarrow & & \swarrow {\scriptstyle g} & \\
B\Gamma_p^{\wedge} & & & &
\end{array}
$$

On the other hand, using the same argument as in the proof of Theorem 4.2, we see that the composite

$$BH \to BG_p^\wedge \to D \to P_{\Sigma B\mathbb{Z}/p}D$$

is null-homotopic because so is its restriction to any generator of H. Apply now Dwyer's version of Zabrodsky's lemma [**14**, Proposition 3.4] to the fibration $BH \to BN_G(S) \to B\Gamma$, taking account of the fact that $P_{\Sigma B\mathbb{Z}/p}D$ is p-complete (which we proved in Proposition 3.2). There exists a map $f : B\Gamma_p^\wedge \to P_{\Sigma B\mathbb{Z}/p}D$ making the following diagram commutative (up to unpointed homotopy).

$$\begin{array}{ccc} BG_p^\wedge & \xrightarrow{\ l \circ h\ } & P_{\Sigma B\mathbb{Z}/p}D \\ {\scriptstyle B\pi_p^\wedge}\downarrow & \nearrow{\scriptstyle f} & \\ B\Gamma_p^\wedge & & \end{array}$$

Let us check that both composites $f \circ g$ and $g \circ f$ are homotopic to the identity. For the first one we only need to check that $f \circ g'$ is homotopic to the coaugmentation map l by the universal property of the nullification.

First, observe that since $P_{\Sigma B\mathbb{Z}/p}D$ is simply connected, it is enough to prove that l and $f \circ g'$ are homotopic in the unpointed category. Now, it is a consequence of the Puppe sequence of the cofibration $BG_p^\wedge \xrightarrow{h} D \to \vee\Sigma B\mathbb{Z}/p$ that two maps from D are homotopic if they are so when pre-composing with h. In our case $f \circ g' \circ h \simeq l \circ h$ by commutativity of the above diagrams.

For the second one, we might as well prove that $g \circ f \circ B\pi_p^\wedge$ is homotopic to $B\pi_p^\wedge$, because of the universal property of the quotient. The same argument as above applies since $B\Gamma_p^\wedge$ is simply connected and we conclude by the commutativity of the diagrams that $g \circ f \circ B\pi_p^\wedge \simeq g \circ p \circ h \simeq B\pi_p^\wedge$. $\qquad\square$

THEOREM 5.6. *Let G be a finite group generated by elements of order p and S a Sylow p-subgroup. Assume that the normalizer $N_G(S)$ controls the fusion in G. Then $\mathrm{CW}_{B\mathbb{Z}/p}BG$ fits in a fibration*

$$\mathrm{CW}_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow B\Gamma_p^\wedge \times \prod_{q \neq p} BG_q^\wedge,$$

where $\Gamma = N_G(S)/\mathrm{Cl}(S)$.

Proof. This is an immediate consequence of the previous proposition, taking account of the description of $P_{\Sigma B\mathbb{Z}/p}D$ given in Proposition 3.2. $\qquad\square$

In the following corollaries we give finally some explicit calculations of the homotopy type of the $B\mathbb{Z}/p$-cellularization, in the case where p-torsion appears in the higher homotopy groups.

COROLLARY 5.7. *Let G be a finite simple group. Then $\mathrm{CW}_{B\mathbb{Z}/2}BG$ has 2-torsion in an infinite number of homotopy groups if and only if G is either a Suzuki group $\mathrm{Sz}(2^n)$ or a unitary group $U_3(2^n)$.*

Proof. From Foote's classification [**19**, Corollary 1] we see that the Suzuki and unitary groups $U_3(2^n)$ are the only simple groups having a proper strongly closed subgroup. Theorem 4.2 implies thus that they are the only candidates for $\mathrm{CW}_{B\mathbb{Z}/2}BG$ to have infinitely many homotopy groups with 2-torsion. Theorem 5.6 implies that 2-torsion appears indeed. $\qquad\square$

In the only two cases where 2-torsion appears, Theorem 5.6 gives the description of the universal cover.

COROLLARY 5.8. *We have two fibrations*

$$\mathrm{CW}_{B\mathbb{Z}/2}B\mathrm{Sz}(2^n) \to B\mathrm{Sz}(2^n) \to B\big((\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1)\big)^{\wedge}_2 \times \prod_{q\neq 2} B\mathrm{Sz}(2^n)^{\wedge}_q,$$

$$\mathrm{CW}_{B\mathbb{Z}/2}BU_3(2^n) \to BU_3(2^n) \to B\big((\mathbb{Z}/2)^{2n} \rtimes \mathbb{Z}/(2^{2n}-1)d^{-1}\big)^{\wedge}_2 \times \prod_{q\neq 2} BU_3(2^n)^{\wedge}_q,$$

where $d = 1$ if n is even and $d = 3$ if n is odd.

For odd primes we are not aware of a classification of the simple groups having a proper strongly closed subgroup. We offer as an illustration the $B\mathbb{Z}/p$-cellularization of the special linear groups described in Example 5.4.

COROLLARY 5.9. *Let p be an odd prime and q be any integer of the form $mp^k + 1$ with $k \geqslant 2$. Then the $B\mathbb{Z}/p$-cellularization of $B\mathrm{PSL}_2(q)$ has p-torsion in an infinite number of homotopy groups and is given by the following fibration:*

$$\mathrm{CW}_{B\mathbb{Z}/p}B\mathrm{PSL}_2(q) \to B\mathrm{PSL}_2(q) \to B\big(\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/2\big)^{\wedge}_p \times \prod_{r\neq p} B\mathrm{PSL}_2(q)^{\wedge}_r.$$

6. Representations

In this last section we construct explicit representations of the Suzuki groups and the special linear groups into certain unitary groups. We think that they might be of independent interest, and they also give an alternative proof of the presence of p-torsion in the $B\mathbb{Z}/p$-cellularization of the classifying spaces. The map we are looking for is best understood as a fusion preserving representation of the Sylow p-subgroup of G in some compact Lie group (this relationship is well explained by Jackson in [23]). This observation definitively shifts the problem to the study of fusion systems, a notion which has recently led Broto, Levi, and Oliver to the concept of p-local finite groups in a topological context (see [8, 9]).

For the Suzuki groups for example we construct below a map $B\mathrm{Sz}(2^n) \longrightarrow BU(2^n-1)^{\wedge}_2$ such that the composition $B\mathbb{Z}/2 \longrightarrow B\mathrm{Sz}(2^n) \longrightarrow BU(2^n-1)^{\wedge}_2$ is null-homotopic for every cyclic subgroup $\mathbb{Z}/2$ in $\mathrm{Sz}(2^n)$. The loop space $\Omega BU(2^n-1)^{\wedge}_2$ is $B\mathbb{Z}/2$-null by [26, Lemma 9.9]. The existence of infinitely many homotopy groups $\pi_n\mathrm{CW}_{B\mathbb{Z}/2}B\mathrm{Sz}(2^n)$ containing 2-torsion is then a direct consequence of Proposition 3.1.

LEMMA 6.1. *Consider the semi-direct product $(\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1)$, where a generator ϕ of $\mathbb{Z}/(2^n-1)$ acts on the elementary abelian group of rank n by permuting transitively the 2^n-1 non-trivial elements. There exists then a faithful representation $\sigma : (\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1) \hookrightarrow U(2^n-1)$.*

Proof. We note that the semi-direct product $(\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1)$ is the affine group $\mathrm{Aff}(2, 2^n)$. The representation can be induced from the sign representation on the subgroup $(\mathbb{Z}/2)^n$, where we see the generators as transpositions. Let us be very explicit in the simplest case, $n = 3$. Send the first generator of the elementary abelian group to the diagonal matrix with entries $(-1, 1, -1, -1, -1, 1, 1)$ and the other elements to the cyclic permutations of it. The standard cyclic permutation matrix of order 7 in $U(7)$ is the image of ϕ and so σ is well defined. $\square$

PROPOSITION 6.2. *There exists a non-trivial map $B\mathrm{Sz}(2^n) \longrightarrow BU(2^n-1)^{\wedge}_2$ such that the composition $B\mathbb{Z}/2 \longrightarrow B\mathrm{Sz}(2^n) \longrightarrow BU(2^n-1)^{\wedge}_2$ is null-homotopic for every cyclic subgroup $\mathbb{Z}/2$ in $\mathrm{Sz}(2^n)$.*

Proof. We construct actually a map from the 2-completion of $B\mathrm{Sz}(2^n)$ and the desired map is then obtained by pre-composing with the completion map $B\mathrm{Sz}(2^n) \to B\mathrm{Sz}(2^n)_2^\wedge$. Since the normalizer of a Sylow 2-subgroup controls fusion in the Suzuki groups, we only need to construct a map out of $B\big(S \rtimes \mathbb{Z}/(2^n-1)\big)_2^\wedge$, where S denotes the 2-Sylow subgroup of $\mathrm{Sz}(2^n)$.

Because $BU(2^n-1)$ is simply connected the sets of homotopy classes of pointed and unpointed maps into $BU(2^n-1)$ agree. The fusion system of the Suzuki group (that is, that of the semi-direct product) is reduced to the Sylow subgroup, so that the set of $[B\mathrm{Sz}(2^n), BU(2^n-1)]$ is isomorphic to the set of fusion preserving representations of S inside $U(2^n-1)$. This means that for finding a non-trivial map $f : B\mathrm{Sz}(2^n) \to BU(2^n-1)_2^\wedge$ it is enough to find a representation $\rho : S \rtimes \mathbb{Z}/(2^n-1) \to U(2^n-1)$ which is non-trivial when restricted to S (compare Dwyer's result [**14**, Theorem 1.4] which we have already used).

On the other hand we want the map f to be null-homotopic when restricted to any cyclic subgroup $\mathbb{Z}/2$. In other words the composite $\Omega_1(S) \to S \rtimes \mathbb{Z}/(2^n-1) \to U(2^n-1)$ should be trivial. The subgroup $\Omega_1(S)$ generated by all elements of order 2 is its center, an elementary abelian subgroup of rank n. It is normal in $S \rtimes \mathbb{Z}/(2^n-1)$, and the quotient is isomorphic to $(\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1)$, where the action of the generator ϕ of order 2^n-1 permutes transitively the non-trivial elements (one has a bijection between these non-trivial classes in the quotient and the squares of their representatives in the center of S). Hence we can define ρ as the composite

$$S \rtimes \mathbb{Z}/(2^n-1) \longrightarrow (\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n-1) \xrightarrow{\ \sigma\ } U(2^n-1),$$

where σ is the representation constructed in the preceding lemma. It is convenient to remark here that the existence of such a representation does not contradict the theorems in this article since $S \rtimes \mathbb{Z}/(2^n-1)$ is not generated by order 2 elements.

Observe that the morphism $B\big(S \rtimes \mathbb{Z}/(2^n-1)\big)_2^\wedge \longrightarrow BU(2^n-1)_2^\wedge$ induced by the representation we have just constructed is clearly trivial when pre-composing with any map $B\mathbb{Z}/2 \longrightarrow B\big(S \rtimes \mathbb{Z}/(2^n-1)\big)_2^\wedge$, because the (unpointed) homotopy classes of these last maps can be identified with the conjugacy classes of $\mathbb{Z}/2$ inside $S \rtimes \mathbb{Z}/(2^n-1)$. $\qquad\square$

REMARK 6.3. We point out that the representation ρ constructed in the previous proposition cannot be induced by a homomorphism $\mathrm{Sz}(2^n) \longrightarrow U(2^n-1)$, not even composed with an Adams operation, because the group $\mathrm{Sz}(2^n)$ is simple, and hence generated by order 2 elements. If it were so, then the homomorphism would be zero over the generators, and thus trivial. Thus our example could be compared with the map $BM_{12} \to BG_2$ constructed by Benson and Wilkerson in [**3**]. It is also related to work of Mislin and Thomas [**27**], and more recently of Broto and Møller [**7**].

The above methods can also be used to obtain interesting representations of the groups $U_3(2^n)$. We now indicate similar results at odd primes for the groups $\mathrm{PSL}_2(q)$ introduced in Example 5.4.

PROPOSITION 6.4. *Let p be an odd prime, $n \geqslant 2$, and $q = p^n$. There exists a non-trivial map $B\mathrm{PSL}_2(q) \longrightarrow BU(2)_p^\wedge$ such that, for every cyclic subgroup $\mathbb{Z}/p$ in $\mathrm{PSL}_2(q)$, the composition $B\mathbb{Z}/p \longrightarrow B\mathrm{PSL}_2(q) \longrightarrow BU(2)_q^\wedge$ is null-homotopic.*

Proof. We first construct an inclusion $j : \mathbb{Z}/p \rtimes \mathbb{Z}/2 \hookrightarrow U(2)$ as follows. If $x = e^{2\pi i/p}$ then the image of the generator of order p is the diagonal matrix with entries $(x, \bar{x})$ and the image of the generator of order 2 is the standard permutation matrix. One can then take for example

the composition

$$f : \mathbb{Z}/p^n \rtimes \mathbb{Z}/2 \longrightarrow \mathbb{Z}/p \rtimes \mathbb{Z}/2 \xrightarrow{\ j\ } U(2),$$

where the first map is the natural projection. Now the induced map at the level of p-completed classifying spaces

$$B\mathrm{PSL}_2(q)_p^\wedge \simeq B(\mathbb{Z}/p^n \rtimes \mathbb{Z}/2)_p^\wedge \longrightarrow BU(2)_p^\wedge$$

is essential, but homotopically trivial when pre-composing with any map from $B\mathbb{Z}/p$.   □

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