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Conditionally flat functors on spaces and groups

Emmanuel Dror Farjoun · Jérôme Scherer

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Abstract Consider a fibration sequence $F \to E \to B$ of topological spaces which is preserved as such by some functor L, so that $LF \to LE \to LB$ is again a fibration sequence. Pull the fibration back along an arbitrary map $X \to B$ into the base space. Does the pullback fibration enjoy the same property? For most functors this is not to be expected, and we concentrate mostly on homotopical localization functors. We prove that the only homotopical localization functors which behave well under pull-backs are nullifications. The same question makes sense in other categories. We are interested in groups and how localization functors behave with respect to group extensions. We prove that group theoretical nullification functors behave nicely, and so do all epireflections arising from a variety of groups.

Keywords Localization · flatness · fiberwise localization · variety of groups

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Introduction

This work originates in the following question. Consider a fibration $F \to E \to B$ of topological spaces and pull it back along a map $X \to B$. Which are the properties of the original fibration that are inherited by the pull-back fibration? For example, if the first fibration has a section, or induces a fibration on the n-th Postnikov stage, then so would one obtained by a pull-back along any map.

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Emmanuel Dror Farjoun
Einstein Institute of Mathematics, Hebrew University of Jerusalem (Gig'at Ram), Jerusalem 91904, Israel. E-mail: farjoun@math.huji.ac.il

Jérôme Scherer
EPFL
SB MATHGEOM, Station 8, MA B3 455, CH-1015 Lausanne, Switzerland. E-mail: jerome.scherer@epfl.ch

Given a functor L, we are interested in the "flatness" property of a fibration $F \to E \to B$ namely, that of being preserved as such by this functor L. So we say that a fibration sequence $F \to E \to B$ over a connected base space is *L-flat* if the sequence $LF \to LE \to LB$ is also a homotopy fibration sequence. A classical example is the fibre lemma of Bousfield and Kan in [5]. This lemma asserts the preservation of principal fibrations with a connected fibre by the homological completion functors R_∞, for any commutative ring R. To which extent localization functors preserve principal fibrations is also the subject of [11]. More generally, Bousfield, in [4], the first author and Smith, in [10], analyzed the "error term" calculating the failure of flatness. Whether L-flatness is preserved under pull-backs was considered to some extent in [1]. A functor L is said to be *conditionally flat* if any pull-back of an L-flat fibration is again L-flat. One direction of the following result has been shown in [1], and many ideas used here, such as fiberwise localization, are explicitly or implicitly present in that article.

Theorem 2.1. *A homotopy localization functor L is conditionally flat if and only if L is a nullification functor P_A for some space A.*

The nullification functor P_A kills A, namely $P_A(A) \simeq *$ and all spaces constructed from A by push-outs, wedges, telescopes, and extensions by fibrations, [9]. Typical examples are Postnikov sections and Quillen's plus-construction, [13].

Thus, for topological spaces, conditional flatness characterizes nullification functors among localization functors. So does right properness. The relationship between right properness and fiberwise localization is explicit in [21], where Wendt works with simplicial sheaves. In this context nullification functors yield right proper left Bousfield localizations, but the converse is not clear.

We then turn to group theory. We replace fibration sequences by short exact sequences $1 \to K \to E \to G \to 1$ and consider again flatness of localization functors in relation to pullbacks along group homomorphisms $H \to G$. It turns out that the situation is more interesting since there are localization functors which are not nullification functors for which L-flatness is preserved by pullbacks of short exact sequences.

This is easily seen to be the case for any right exact functor such as the abelianization functor $G \to G_{ab}$. In fact, consider the quotient $L_c G = G/\Gamma_c(G)$ by the c-th term in the lower central series, turning a group G into a nilpotent one of some fixed class c. Flatness with respect to these functorial quotients is a property which behaves well with taking pull-backs. They are examples of localization functors associated to a variety, [17], a notion we recall shortly in Section 3.

Theorem 3.2. *Let $\mathcal{W}$ be any variety of groups. The associated localization functor L in the category of groups is then right exact and thus conditionally flat.*

In the terminology of Bourn and Gran, [2], conditionally flat localization functors in the category of groups correspond to the so-called *fibered reflections.* To look for a characterization of conditionally flat functors in other semi-abelian categories is also a question of interest, [12]. In this work we only deal with spaces and groups. Starting from a short exact sequence of groups, the classifying space construction yields a fibration of spaces. The question of conditional flatness translates then into the homotopy category. Theorem 3.2 shows thus that there are homotopical localization functors which are not nullification functors, but preserve nevertheless

L-flat fibration sequences of the form $BK \to BE \to BG$ under certain pull-backs. This is shortly discussed in Remark 3.1.

Organization and content. The rest of the paper is organized as follows: The first section gives basic definitions and notations. The second contains the main result about fibration sequences and their conditional preservation by functors. In the third section flatness of functors on groups is considered, here right exact functors are shown to be conditionally flat. The last section deals with (counter-)examples and possible further developments.

1 Notation and terminology

We are interested in properties of fibration sequences of pointed spaces (or simplicial sets) and extensions of groups. As they share many common features we will introduce some terminology which applies to both settings.

We will work with homotopy localization functors L in the sense of Bousfield, see [3] and [13], in the category of pointed spaces or groups. In practice we fix a map f of spaces or groups and consider the localization functor L_f which "inverts the map f", [13]. A space Z is *f-local* if $\mathrm{map}(f, Z)$ is a weak equivalence and a map g is an *f-equivalence* if $\mathrm{map}(g, Z)$ is a weak equivalence for any f-local space Z. The co-augmented functor L_f is characterized by the fact that the natural co-augmentation map $\eta_X : X \to L_f X$ is an f-equivalence and $L_f X$ is f-local. A space X is called *L_f-acyclic* if $L_f X$ is contractible. Historically, the motivating example is homological localization as follows.

Example 1.1 Bousfield showed that there exists a universal ordinary integral homology equivalence f so that L_f is homological localization $L_{H\mathbb{Z}}$. The natural map $X \to L_{H\mathbb{Z}}X$ is a homology equivalence for any space X. In particular, $L_{H\mathbb{Z}}X$ is contractible for any $H\mathbb{Z}$-acyclic space X.

Instead of defining L-flatness only for fibrations as we did in the introduction for simplicity, we do it for any map.

Definition 1.1 Let L be a homotopy functor on pointed spaces. A map $E \to B$ is *L-flat* if the canonical comparison map $LFib(E \to B) \to Fib(LE \to LB)$ is a weak equivalence.

This definition makes sense not only for localization functors, but arbitrary endofunctors (such as cellularization functors for example, but possibly also non-idempotent ones like the James construction $JX \simeq \Omega\Sigma X$, or the infinite symmetric product $SP^\infty X$).

Rather than checking if an arbitrary map is L-flat, it is convenient to replace it by a fibration $E \to B$. Definition 1.1 now reads as follows: A fibration sequence $F \to E \to B$ is L-flat if and only if $LF \to LE \to LB$ is again a fibration sequence. This form of L-flatness is suitable for a translation in the category of groups. Hence, a group extension $1 \to N \to E \to G \to 1$ is L-flat if $1 \to LN \to LE \to LG \to 1$ is again an extension of groups.

Example 1.2 Let P be a nullification functor, i.e. $P = L_f$ for $f : A \to *$. Then any map $E \to B$ over a P-local base space B is P-flat, [13, Corollary D.3].

Example 1.3 We choose for f the map collapsing a sphere S^{n+1} to a point, so that the localization $X \to L_f X = P_{S^{n+1}} X$ is homotopy equivalent to taking the n-th Postnikov stage $X \to X[n]$. Here, a fibration sequence $F \to E \to B$ is "n-Postnikov-flat" if and only if the connecting homomorphism $\pi_{n+1} B \to \pi_n F$ is trivial. Therefore, if $F \to E \to B$ is n-Postnikov-flat, so is any pullback fibration sequence $F \to E \times_B X \to X$, for any map $X \to B$.

A typical example of a fibration sequence which is not n-Postnikov flat is the path-loop fibration on a space which is not $(n+1)$-connected such as S^{n+1}. The fibration $\Omega S^{n+1} \to PS^{n+1} \to S^{n+1}$ is not n-Postnikov flat, since applying Postnikov sections destroys the exactness of the sequence of homotopy groups of the spaces involved.

Most maps are not L-flat for a given localization functor L. The question we ask is about the preservation of flatness under base change, that is, if we happen to work with an L-flat map, we ask whether the map obtained by pulling back along an arbitrary map to the base is L-flat again.

Definition 1.2 The map $E \to B$ is *fully L-flat* if all its pullbacks are L-flat. A functor L is *conditionally flat* if any pull back of an L-flat map is again L-flat, i.e. if any L-flat map is fully L-flat.

Thus full L-flatness refers always to a map and means both its L-flatness (because one can choose to pull-back along the identity map) and the L-flatness of any of its pullbacks. Namely, for any map $B' \to B$ the map $E \times_B B' \to B'$ is L-flat. As mentioned in the introduction, the fibre lemma of Bousfield and Kan states that all principal fibrations with a connected fibre-group are R_∞-fully flat.

The main players here are not the various maps we consider but rather the functors L. Conditional flatness refers to functors: In general, localization functors are not flat, i.e. they do not preserve all fibration sequences but often, if a map is L-flat, then so is any pull-back. This is a property of the functor L which we call here "conditionally flat."

We will use the same terminology for group extensions and group theoretic localization functors.

2 Localization of fibration sequences

We extend in this section the results of Berrick and the first author in [1]. It was shown there that nullification functors are always conditionally flat and we prove now that, in fact, a homotopy localization functor L is conditionally flat if and only if it is a nullification functor (such as the Postnikov sections described in Example 1.3). Parts of our arguments resemble those in [1], but we prefer to include a complete proof because we will follow the precise same steps in the next section for group theoretic localizations.

Theorem 2.1 *A homotopy localization functor L is conditionally flat if and only if L is a nullification functor P_A for some space A.*

The proof will be given at the end of the section. We start with a few reduction steps. The first one allows us to work with maps having local homotopy fibers.

Lemma 2.1 *Let L be a homotopy localization functor and assume that any L-flat map with L-local homotopy fiber is fully L-flat. Then L is conditionally flat.*

Proof Let us consider a pull-back diagram of fibration sequences

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & X \arrow[d] \\
F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

where the bottom map $E \to B$ is L-flat. We have to show that so is the top map $P \to X$. Applying fiberwise localization, [13, Section 1.F] to both fibrations yields a new diagram of horizontal fibration sequences

$$\begin{tikzcd}
LF \arrow[r] \arrow[d, equal] & \overline{P} \arrow[r] \arrow[d] & X \arrow[d] \\
LF \arrow[r] & \overline{E} \arrow[r] & B
\end{tikzcd}$$

together with maps $E \to \overline{E}$ and $P \to \overline{P}$ which are L-local equivalences. Notice that $\overline{P}$ is obtained as the homotopy pull-back of $\overline{E} \to B \leftarrow X$. By assumption the map $E \to B$ is L-flat, thus so is the fiberwise localization $\overline{E} \to B$, since applying L to it yields the map $LE \to LB$, whose homotopy fiber is LF. We suppose that this property is preserved by taking pull-backs of fibrations with local fiber. Therefore we conclude that the map $\overline{P} \to X$ is L-flat, which implies in turn that $P \to X$ is so. □

The second step reduces the problem to studying fibration sequences of local spaces.

Lemma 2.2 *Let L be a localization functor and assume that all fibration sequences of L-local spaces are fully L-flat. Then L is conditionally flat.*

Proof We know from the previous lemma that we can assume the fiber to be L-local. We consider thus an L-flat fibration sequence $F \to E \to B$ where F is L-local and a map $g : X \to B$. We can also assume that B is connected. We complete it to the following diagram by constructing first the pullback along g and second, by localizing the bottom row:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
F \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & B \arrow[d] \\
F \arrow[r] & LE \arrow[r] & LB
\end{tikzcd}$$

Since B, and hence LB are connected spaces, we see that E is the homotopy pull-back of the diagram $LE \to LB \leftarrow B$, and therefore P is the homotopy pull-back of $LE \to LB \leftarrow X$. We conclude that the top fibration sequence is L-flat. □

Our third and last step allows us to perform the pullback construction along a very specific map, namely the localization map $\eta_X : X \to LX$.

Lemma 2.3 *Let L be a localization functor and assume that, for any connected space X and any fibration sequence $F \to E \to LX$ of L-local spaces, the pullback fibration sequence along $\eta_X : X \to LX$ is L-flat. Then L is conditionally flat.*

Proof We need only prove by Lemma 2.2 that a fibration sequence $F \to E \to B$ of L-local spaces is fully L-flat. Consider thus any map $g : X \to B$. We must show that the pull-back fibration sequence $F \to P \to X$ is L-flat. Since g factors through the localization map $X \to LX$ we construct a diagram of fibration sequences involving the L-local homotopy pull-back Q of $E \to B \leftarrow LX$:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & X \arrow[d, "\eta_X"'] \arrow[dd, bend left, "g"] \\
F \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & LX \arrow[d] \\
F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

Since by our assumptions the space Q is a homotopy pull back of local spaces, it follows that the top right square is also a homotopy pull-back square and the middle row is a fibration sequence of L-local spaces, [13, A.8 (e3)]. By assumption the top fibration is preserved by L. □

We are now ready to prove the main theorem of this section.

Proof (Proof of Theorem 2.1) We consider a fibration sequence $F \to E \to LB$ of L-local spaces and will show that the pullback along the localization map $\eta_B : B \to LB$ is L-flat. We will deduce from Lemma 2.3 that L is conditionally flat. Let us observe the following diagram:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & B \arrow[d] \\
F \arrow[r] & E \arrow[r] & LB
\end{tikzcd}$$

If L is not a nullification functor, there exists a space B such that the homotopy fiber $\overline{L}B$ of the localization map $B \to LB$ is not L-acyclic. An explicit example is constructed in [1, Theorem 2.1, (iv) $\Rightarrow$ (i)]. However the fibration sequence $\Omega(LB) \to P(LB) \to LB$ is one of L-local spaces. The pull-back fibration sequence $\Omega(LB) \to \overline{L}B \to B$ is not L-flat because the localization of the total space $L\overline{L}B$ is not contractible.

When L is of the form P_A, the homotopy fiber of the localization map $B \to P_A B$ is P_A-acyclic. Therefore the fibration sequence $\overline{P}_A B \to Q \to E$ is preserved by P_A, [13, Theorem 1.H.1], i.e. $P_A Q \simeq E$ which means that the fibration sequence $F \to Q \to B$ is P_A-flat. □

Hence, nullification functors such as plus-constructions, Postnikov sections, $B\mathbb{Z}/p$-nullification appearing in Miller's work on the Sullivan conjecture, [16], are all conditionally flat. Counter-examples can now also be easily constructed.

Example 2.1 Consider the localization functor $L_{H\mathbf{Z}}$ with respect to ordinary homology $H^*(-;\mathbb{Z})$. There are many spaces for which the homotopy fiber of the localization are not acyclic, often not even connected. One of the "smallest" examples is Whitehead's example, [22, IV.7 Example 3], of a three cell complex $X = (S^1 \vee S^2) \cup e^3$ having the homology of a circle. The homological localization map $X \to S^1$ coincides with the first Postnikov section, so that the homotopy fiber is the universal cover $\tilde{X}$, a simply connected but non-trivial $H\mathbb{Z}$-local space. The pull-back of the path-loop fibration $\mathbb{Z} \to PS^1 \to S^1$ along the map $X \to S^1$ yields a fibration $\mathbb{Z} \to \tilde{X} \to X$ which is not $L_{H\mathbb{Z}}$-flat.

3 Conditionally flat group-functors and varieties

We move now to the category of groups, replacing the notion of fibration sequence by short exact sequences. Our aim is to look at the notions of flatness and conditional flatness for functors and extensions of groups. A localization functor in the category of groups is a co-augmented idempotent functor L. The terminology and notation are the same as for homotopical localization. In particular, any group homomorphism $\varphi : A \to B$ determines a localization functor L_φ such that the natural map $G \to L_\varphi G$ is a φ-equivalence to a φ-local group (so $\varphi^* : \mathrm{Hom}(B, L_\varphi G) \to \mathrm{Hom}(A, L_\varphi G)$ is an isomorphism).

We start this section by proving that group theoretical nullification functors, that is, those localization functors L_φ corresponding to homomorphisms φ of the form $A \to 1$, are conditionally flat, even though they are not the only ones. Such a nullification functor is written P_A. We notice that the same reduction steps we went through for spaces in Section 2 do work for groups.

Proposition 3.1 *Let L be a localization functor in the category of groups. Assume that, for any group G and any extension of L-local groups $K \to E \to LG$, the pull-back along the localization morphism $\eta_G : G \to LG$ is L-flat. Then L is conditionally flat.*

Proof We must show that the pull-back of an L-flat extension is in turn L-flat. The first reduction step allowing us to consider only extensions with local kernel is obtained by applying fiberwise localization to our group extensions. Such a construction is available for groups thank to work of Casacuberta and Descheemaeker, [7]. Thus one can construct, for any group extension $1 \to K \to E \to G \to 1$, another extension $1 \to LK \to \bar{E} \to G \to 1$ where the fiber (the kernel) has been localized and a morphism $E \to \bar{E}$ which is an L-equivalence. The second step reduces to the study of extensions of local groups and this works simply because one recognizes a pull-back square by comparing the kernels. The third and last step is exactly as in Lemma 2.3 and permits us to pull-back along a localization map $G \to LG$.

A group K is *L-acyclic* if LK is the trivial group. We say it is "killed by L".

Theorem 3.1 *Any nullification functor in the category of groups is conditionally flat.*

Proof The kernel of the localization morphism $G \to LG$ is L-acyclic when (in fact if and only if) L is a nullification functor, [19, Proposition 3]. □

We move now as promised to more "exotic" conditionally flat localization functors, that is some which are not nullifications. Our motivation was to study the interplay of pulling back an extension and taking the quotient by the lower central series. We are now ready to come back to this question.

Example 3.1 Assume that the group extension $1 \to K \to E \to G \to 1$ *abelianizes well*, that is, the abelianization $0 \to K_{ab} \to E_{ab} \to G_{ab} \to 0$ forms an extension (of abelian groups). Then for any morphism $H \to G$, the pull-back extension $K \to P \to H$ also abelianizes well. In our general terminology, abelianization is conditionally flat.

The argument is simple. Abelianization is right exact, a fact that can be proved either directly, or by using the group homology five term exact sequence, see [20, 6.8.3] or deduce it from Hopf's formula, [6, Exercise II.5.6]. Hence we only need to show that $K_{ab} \to P_{ab}$ is injective. The extension $1 \to K \to E \to G \to 1$ is flat for abelianization by assumption, hence $K_{ab} \to E_{ab}$ is injective. As it factors through P_{ab} the conclusion follows.

The same proof actually applies to *any* right exact functor.

Proposition 3.2 *Let F be a right exact functor in the category of groups. Then F is conditionally flat.* □

A *variety of groups* $\mathcal{W}$ is defined by a set of words W in a free group F on a countable, infinite set of generators $\{x_1, x_2, x_3, \dots\}$. Following [17], $\mathcal{W}$ is the family of all groups G having the property that every homomorphism from F to G sends the words in W to 1. Take $\phi : F \to F/WF$, where WF is the normal subgroup generated by images of words in W under all homomorphisms $F \to F$. The localization functor L_ϕ sends then a group G to the largest quotient which belongs to the variety $\mathcal{W}$, [8, Proposition 3.1]. The kernel can be described as the normal subgroup WG of G generated by all images of words in W under homomorphisms from F.

Example 3.2 For any given integer $c \geq 1$, we take W to be generated by the single word $[\dots[x_1, x_2], \dots x_c], x_{c+1}]$, a c-fold commutator. For any group G the subgroup WG is nothing but $\Gamma_c(G)$ the c-th term in the lower central series. Thus, the localization L_ϕ sends G to $G/\Gamma_c(G)$. When $c = 1$ for example, W is generated by a single commutator $[x_1, x_2]$. A group belongs to $\mathcal{W}$ if and only if it is abelian, the group homomorphism ϕ is $F \to F/[F, F] = F_{ab}$ and L_ϕ is abelianization.

In general $W(WG) \neq WG$, as is shown by abelianization (of the dihedral group of order 8 say). In fact Casacuberta, Rodríguez and Scevenels show that $W(-)$ is idempotent if and only if the corresponding localization is a nullification, [8, Theorem 2.3]. We prove now that any variety of groups determines a right exact localization functor, hence a conditionally flat functor. We could also have applied our general principle Proposition 3.1 and proven "by hand" that the pull-back of an extension of local groups $K \to E \to G/WG$ along the localization map $G \to G/WG$ is flat.

Theorem 3.2 *Let $\mathcal{W}$ be any variety of groups. The associated localization functor L in the category of groups is then right exact and thus conditionally flat.*

Proof Let W be the set of words defining $\mathcal{W}$. By Proposition 3.2 it is enough to prove that for any extension $1 \to K \to E \xrightarrow{p} G \to 1$, the sequence $K/WK \to E/WE \to G/WG \to 1$ is exact. The localization $G \to G/WG$ is surjective, hence so is $\bar{p} : E/WE \to G/WG$. We only need to identify the classes of the form eWE, for $e \in E$, in the kernel of this last morphism $\bar{p}$, which means that $p(e) \in WG$. In other words $p(e)$ can be written as a product γ of conjugates of words $w(g_i)$ with $w \in W$. Since p is surjective there is a product ϵ of conjugates of the same words $w(e_i)$ whose image under p is $p(e) = \gamma$. Therefore, e and ϵ differ by an element k in the kernel K. But now, since $\epsilon \in WE$, we have

$$eWE = e\epsilon^{-1}WE = kWE$$

which proves exactness at E/WE. □

Since nilpotency is described by a variety of groups, we obtain the following result:

Corollary 3.1 *The localization functor in the category of groups taking a group G to its quotient $G/\Gamma_c(G)$ by the lower central series is conditionally flat.* □

Remark 3.1 The classifying space functor B : Groups $\to$ Spaces$_*$ takes a discrete group G to the Eilenberg–Mac Lane space BG. Let φ be a group homomorphism such that the localization functor L_φ is conditionally flat in the category of groups, but it not a nullification (for example the above quotients by a given term of the lower central series). The homotopical localization $L_{B\varphi}$ associated to the corresponding map of classifying spaces is not conditionally flat as we know from Theorem 2.1. However extensions of φ-local groups yield fibration sequences of $B\varphi$-local classifying spaces and pull-back of such fibration sequences along any map of classifying spaces are L-flat.

4 Examples, counter examples, and open questions

Localization functors in the category of groups associated to varieties of groups or nullification functors are conditionally flat. However, as soon as the localization we consider is not one corresponding to a variety, things can easily go “wrong”. Let us construct various counter examples.

4.1 Epireflections and quasi-varieties

A localization functor L in the category of groups is called an *epireflection* if the localization morphism $G \to LG$ is an epimorphism for any group G. Such localization functors are in one to one correspondence with subfunctors of the identity, usually called *radicals*, since to an epireflection one associates the radical R_L defined by $R_L(G) = \mathrm{Ker}(G \to LG)$. A good source for the group theorist’s point of view on radicals is Robinson’s book [18] and epireflections have been studied thoroughly in [19].

A localization functor in the category of groups is an epireflection if and only if there exists an epimorphism φ such that L_φ is L. Thus every variety of groups $\mathcal{W}$

determines an epireflection. In the category of groups we use the notation C_n for the cyclic group of order n (and the group law is multiplication), but in the category of abelian groups, where the group law is addition, we use the notation $\mathbf{Z}/n$.

Theorem 4.1 *There are epireflections L which are not conditionally flat.*

Proof Let $\phi : C_4 \to C_2$ be the projection and choose $L = L_\phi$. Any torsion-free group is local with respect to this epireflection since there are no non-trivial morphism from a torsion group to a torsion-free group. Moreover the cyclic group of order 2 is local as well (it is the localization of C_4).

Therefore the abelian group extension $\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2$ is an extension of local groups. Let us pull it back along φ itself. The pull-back P is an extension of $\mathbf{Z}$ by $\mathbf{Z}/2$, which must be trivial, so P is isomorphic to $\mathbf{Z} \times \mathbf{Z}/2$, another local group! The pull-back extension $\mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}/2 \to \mathbf{Z}/4$ is therefore not preserved by L. □

Remark 4.1 Localization with respect to $C_4 \to C_2$ is an epireflection, and even better a localization associated to a so-called *quasi-variety*, [15]. Whereas for a variety one imposes certain words to become trivial, in a quasi-variety one does so provided certain equations are satisfied. In the previous proof the condition is that $x^4 = 1$. If so, then one imposes $x^2 = 1$. We have thus actually proven a little bit more than stated in Theorem 4.1: There are epireflections associated to quasi-varieties which are not conditionally flat.

4.2 Other localization functors

We turn now to a general localization functor L and study which are its features which prevent L from being conditionally flat. What is the general principle which lies behind this compatibility between pulling back and localizing? Since nullification functors are known to be conditionally flat, we discard them and work from now on with a localization functor which is not of the form P_A.

Lemma 4.1 *Let L_ϕ be a localization functor which is not a nullification. Then there exists a non-identity localization morphism $G \to L_\phi G$ which has L_ϕ-local kernel.*

Proof Let $L_{\mathcal{E}(\phi)}$ be the universal epireflection associated to L_ϕ, [19, Theorem 8], which means that the localization morphism $G \to L_\phi G$ factors as

$$G \twoheadrightarrow L_{\mathcal{E}(\phi)} G \hookrightarrow L_\phi G$$

As L_ϕ is not a nullification functor by assumption, we have to deal with two cases. In the first one, the epireflection is a nullification, and then there exists a $\mathcal{E}(\phi)$-local group G such that $G \to L_\phi G$ is injective, hence has a local kernel. In the second one the epireflection is not a nullification, i.e. there exists a group G such that the kernel of $G \twoheadrightarrow L_{\mathcal{E}(\phi)} G$ is not acyclic, [8]. Fiberwise localization then yields a morphism $\overline{G} \to L_\phi G$ with (non-trivial) local kernel. □

The previous lemma justifies the choice of G in the following proposition.

Proposition 4.1 *Let $f : A \to B$ be a group homomorphism and let $L = L_f$. Assume that there exist a non-identity localization morphism $G \to LG$ with local kernel and a surjection $E \to LG$ from a local group E such that* $\mathrm{Hom}(A, E) = \{1\} = \mathrm{Hom}(B, E)$. *Then the pull-back P of the diagram $E \to LG \leftarrow G$ is local. In particular $P \to E$ is not the localization morphism and L is not conditionally flat.*

Proof Any morphism from A, respectively B, to P is given by a pair of compatible morphisms to G and E. By assumption the morphism to E is trivial so that the morphism to G must factorize through the kernel of the localization, which is local. Therefore $\mathrm{Hom}(A, P) = \{1\} = \mathrm{Hom}(B, P)$. □

This construction helps to find many localization functors which are not conditionally flat. The first occurence of such a localization was the epireflection associated to a quasi-variety encountered in the proof of Theorem 4.1.

Example 4.1 Let $f : A_n \hookrightarrow A_{n+1}$ be the standard inclusion. It has been shown by Libman, [14, Example 3.4], that it is a localization for $n \geq 7$. Pick a (free) presentation $F_1 \to F_0 \to A_{n+1}$. Any free group is obviously f-local, so that the proposition applies and the pull-back of $A_n \to A_{n+1} \leftarrow F_0$ is also f-local.

4.3 What about non-localization functors?

If we consider a functor which behaves well with push-outs and more generally colimits, and (therefore) badly with respect to extensions and pull-backs, we will see that it has very little chance to be conditionally flat. For any group G we write $S_p(G) \subset G$ for the subgroup generated by its elements of order p.

Proposition 4.2 *The functor S_p is not conditionally flat.*

Proof We exhibit a counter-example: Set $p = 2$ and consider the (central) extension $C_2 \to D_8 \to C_2 \times C_2$ where D_8 is the dihedral group of order 8, given by the standard presentation $< x, y \,|\, x^4 = y^2 = 1 = yxyx >$. This extension is S_2-flat since D_8 is generated by y and yx, both elements of order 2. However if we pick in the base $C_2 \times C_2$ the copy of C_2 generated by the image of x and pull the extension back along this inclusion, we get the extension $C_2 \to C_4 \to C_2$ which is not S_2-flat. □

In fact we could also have chosen the analogous property defined by replacing the subgroup $S_p(-)$ by C_p-cellularization. The class of C_p-cellular groups is closed under colimits and the question we ask deals with extensions and pull-backs. This is why we should not expect them to behave well together. One should maybe rather ask the dual question about the interplay of push-outs and cellularization.

4.4 Open questions

We know now that general group localization functors do not behave as nicely as one could expect with respect to pulling back extensions, not even for abelian groups. Nullifications and epireflections coming from group varieties are the only functors we know of that behave well. We have not dealt with localization functors L for which $G \to LG$ is not surjective, such as localization at a set of primes.

Question A. Are there conditionally flat localization functors which are not epireflections?

Notice that rationalization, and localization at a set of primes, are exact functors in the category of abelian groups. They are therefore flat, hence conditionally flat in the category of abelian groups.

Question B. Is rationalization, i.e. localization with respect to multiplication by p on the integers for all prime numbers p, conditionally flat in the category of groups?

By moving from the category of spaces to that of groups, we found more conditionally flat localization functors. By restricting even more to a strict subcategory of groups, the class of conditionally flat functors will increase.

Question C. What does conditional flatness mean in a full subcategory of groups, such as abelian or nilpotent groups?

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