Homotopical algebra
Exercise Set 7
(revised version)

09.04.2018

Exercise 2 is to be handed in on 16.04.2018.

1. Let $K_\bullet$ be a simplicial set. Prove that there is a natural isomorphism of simplicial sets
$$\coprod_{n\geq 0} K_n \times \Delta[n]/\sim \xrightarrow{\cong} K_\bullet,$$
where $(x, Y(\delta^i)(\xi)) \sim (d_i x, \xi)$ and $(x, Y(\sigma^j)(\zeta)) \sim (s_j x, \zeta)$ for all $x \in K_n$, $\xi \in \Delta[n-1]_m$, $\zeta \in \Delta[n+1]_m$, $0 \leq i, j \leq n$, and $m, n \geq 0$. (Here, Y denotes the Yoneda functor from $\mathbf{\Delta}$ to sSet.)

2. In this exercise we study the simplicial analog of the topological mapping space.
   (a) Let $K_\bullet$ and $L_\bullet$ be simplicial sets. Explain how to define the faces and degeneracies of a simplicial set $\mathrm{Map}(K_\bullet, L_\bullet)_\bullet$ with
   $$\mathrm{Map}(K_\bullet, L_\bullet)_n = \mathsf{sSet}(K_\bullet \times \Delta[n], L_\bullet),$$
   using the maps δ^i and σ^j.
   (b) Prove that for any simplicial set $K_\bullet$, the functors $-\times K_\bullet$ and $\mathrm{Map}(K_\bullet, -)$ are adjoint.
   (c) Use (b) and the Yoneda Lemma to prove that there is a natural isomorphism of simplicial sets
   $$\mathrm{Map}\big(J_\bullet, \mathrm{Map}(K_\bullet, L_\bullet)\big) \cong \mathrm{Map}(J_\bullet \times K_\bullet, L_\bullet),$$
   for all simplicial sets $J_\bullet, K_\bullet, L_\bullet$.

3. Show that for all $m, n \geq 0$ and $f \in \mathbf{\Delta}(m, n)$, there exist unique sets of integers
$$n \geq i_1 > \cdots > i_k \geq 0 \quad \text{and} \quad 0 \leq j_1 < \cdots < j_l \leq m$$

such that
$$f = \delta^{i_1} \cdots \delta^{i_k} \sigma^{j_1} \cdots \sigma^{j_l},$$
where $n - k + l = m$.

4. The goal of this exercise is to introduce and study the elementary properties of the *nerve functor*, $\mathsf{N}_\bullet : \mathsf{Cat} \to \mathsf{sSet}$.

   (a) Let Poset denote the category of posets (partially ordered sets), and let Cat denote the category of small categories. Define a functor $\iota : \mathsf{Poset} \to \mathsf{Cat}$ such that $\operatorname{Ob} \iota(P, <) = P$ and that is *faithful*, i.e., injective on morphisms. The functor ι allows us to view any poset, such as the totally ordered set $[n]$, as a category in a natural way.

   (b) Let $\mathsf{N}_\bullet : \mathsf{Cat} \to \mathsf{sSet}$ be the functor defined by
   $$\mathsf{N}_\bullet \mathcal{C} = \mathsf{Cat}\big(\iota(-), \mathcal{C}\big) : \mathbf{\Delta}^{op} \to \mathsf{Set}.$$
   Show that $\mathsf{N}_0 \mathcal{C} = \operatorname{Ob} \mathcal{C}$, while for all $n > 0$,
   $$\mathsf{N}_n \mathcal{C} = \big\{ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n \mid f_i \in \operatorname{Mor} \mathcal{C}\ \forall i \big\}.$$
   Describe explicitly the face maps $d_i : \mathsf{N}_n \mathcal{C} \to \mathsf{N}_{n-1} \mathcal{C}$, the degeneracies $s_j : \mathsf{N}_n \mathcal{C} \to \mathsf{N}_{n+1} \mathcal{C}$, and the simplicial map $\mathsf{N}_\bullet F : \mathsf{N}_\bullet \mathcal{C} \to \mathsf{N}_\bullet \mathcal{D}$ induced by a functor $F : \mathcal{C} \to \mathcal{D}$. What are the nondegenerate simplices of $\mathsf{N}_\bullet \mathcal{C}$?

   (c) Show that $\mathsf{N}_\bullet \iota[n] \cong \Delta[n]$ for all $n \geq 0$.

   (d) Let $\mathcal{B} : \mathsf{Gr} \to \mathsf{Cat}$ denote functor sending a group G to the category $\mathsf{B}G$. The composite functor $\mathsf{B}_\bullet = \mathsf{N}_\bullet \circ \mathcal{B}$ is the *simplicial bar construction*. Calculate $\mathsf{B}_\bullet(\mathbb{Z}/2\mathbb{Z})$. What are its nondegenerate simplices?

   (e) Apply the Yoneda Lemma to proving that the nerve functor induces a bijection
   $$\mathsf{N}_\bullet : \mathsf{Cat}(\mathcal{C}, \mathcal{D}) \to \mathsf{sSet}(\mathsf{N}_\bullet \mathcal{C}, \mathsf{N}_\bullet \mathcal{D})$$
   for all small categories $\mathcal{C}$, $\mathcal{D}$.

5. Prove that there are homeomorphisms $|\Delta[n]| \cong \Delta^n$ and $|\partial \Delta[n]| \cong \partial \Delta^n \cong S^{n-1}$.