Homotopical algebra Exercise Set 5

19.03.2018

Exercise 5 is to be handed in on 26.03.2018.

- 1. Let \emptyset denote the empty category, i.e., the category with an empty set of objects (and thus an empty set of morphisms). Let C be any category. Show that there is a unique functor $\emptyset \to C$. Determine when the (co)limit of this functor exists and, when it does it exist, what its value is.
- 2. Let P denote the category with $Ob P = \{0, 1\}$ and only two non-identity morphisms, $d_0, d_1 \in P(1, 0)$. Let C be any category. The (co)limit of a diagram $F : P \to C$, when it exists, is called an *(co)equalizer*. Provide explicit formulas for equalizers and coequalizers in Set, Top, Ab, and Gr.
- 3. Let G be a group. Let **GSet** denote the category of G-sets, i.e., sets equipped with a G-action and G-equivariant maps between them.
 - (a) Show that GSet and $\mathsf{Set}^{\mathsf{B}G}$ are isomorphic categories.
 - (b) Show that $\operatorname{colim}_{\mathsf{BG}} F$ and $\operatorname{lim}_{\mathsf{BG}} F$ exist for all $F \in \operatorname{Ob} \mathsf{Set}^{\mathsf{BG}}$, by computing them explicitly. (You should recognize them as constructions you already know, associated to any *G*-action on a set.)
- 4. Let C be a bicomplete category, and let $A \in Ob C$. Show that C/A and A/C are both bicomplete.
- 5. Let C be a category. Let V denote the category with $Ob V = \{1, 2, 3\}$ and as only non-identity morphisms $a : 1 \rightarrow 3$ and $b : 2 \rightarrow 3$.
 - (a) The limit of a functor $F : V \to C$, when it exists, is called a *pullback* and denoted $F(1) \times_{F(3)} F(2)$. Show that Set, Top, Ab, and Gr admit pullbacks by providing explicit constructions.
 - (b) The colimit of a functor $F : V^{op} \to C$, when it exists, is called a *pushout* and denoted $F(1) \coprod_{F(3)} F(2)$. Show that Set, Top, Ab, and Gr admit pushouts by providing explicit constructions.

- (c) Let S be a class of morphisms in C. Show that LLP(S) is preserved under pushout, and RLP(S) is preserved under pullback. In other words, if $i : A \to X$ is in LLP(S), then, for any morphism $f : A \to B$, the induced morphism $B \to B \coprod_A X$ into the pushout is also in LLP(S). Dually, if $p : E \to B$ is in RLP(S), then for any morphism $f : A \to B$, the induced morphism $A \times_B E \to A$ out of the pullback is also in RLP(S). (Note that this result implies that Hurewicz cofibrations are preserved under pullback in Top, while Hurewicz fibrations are preserved under pullback in Top.)
- 6. For any small category J and any bicomplete category C, explain how to construct a functor $\lim_{J} : C^{J} \to C$ (respectively, $\operatorname{colim}_{J} : C^{J} \to C$) defined on objects by sending any J-shaped diagram F to an element of the isomorphism class of limits of F (respectively, an element of the isomorphism class of colimits of F).