Homotopical algebra
Exercise Set 4

12.03.2018

Exercise 5 is to be handed in on 19.03.2018.

1. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Show that F *preserves* isomorphisms, i.e., the image under F of an isomorphism in C is an isomorphism in D.

2. Let $F_{\text{Ab}} : \mathsf{Set} \to \mathsf{Ab}$ denote the free abelian group functor and $U : \mathsf{Ab} \to \mathsf{Set}$ the forgetful functor. Let $\tau : F_{\text{Ab}}U \to \text{Id}_{\mathsf{Ab}}$ denote the natural transformation constructed in class.

   (a) Show that there is a natural transformation $\iota : \text{Id}_{\mathsf{Set}} \to UF_{\text{Ab}}$ such that the composite
   $$A \xrightarrow{\iota_A} UF_{\text{Ab}}(A) \xrightarrow{U(\tau_{(A,+,0)})} A$$
   is the identity on the underlying set A of any abelian group $(A, +, 0)$.

   (b) For any set X, show that $F_{\text{Ab}}(X)$ really is free on X, in the sense that for every abelian group $(A, +, 0)$ and every function $f : X \to A$, there is a unique homomorphism $\hat{f} : F_{\text{Ab}}(X) \to (A, +, 0)$ such that $U(\hat{f}) \circ \iota_X = f$.

3. Let C be a category, and let $A \in \operatorname{Ob}\mathsf{C}$. Let J be another category. Let $\Delta_A : \mathsf{J} \to \mathsf{C}$ denote the functor specified on objects by $\Delta_C(j) = A$ for all $j \in \operatorname{Ob}\mathsf{J}$ and on morphisms by $\Delta_C(f) = \text{Id}_A$ for all $f \in \operatorname{Mor}\mathsf{C}$.

   (a) Show that Δ_A is indeed a functor, which implies in particular that functors do not *reflect* isomorphisms, i.e., that the image of a morphism f under a functor is an isomorphism doesn't imply that f is an isomorphism.

   (b) Let $F : \mathsf{J} \to \mathsf{C}$ be another functor. Show that a natural transformation $F \to \Delta_A$ can be interpreted as functor $\mathsf{J} \to \mathsf{C}/A$, while a natural transformation $\Delta_A \to F$ can be interpreted as a functor $\mathsf{J} \to A/\mathsf{C}$.

4. Let $F, G : \mathsf{D} \to \mathsf{C}$ be functors. Let $S, T : \mathsf{C}^{\to} \to \mathsf{C}$ denote the source and target functors seen in class. Show that there is a bijection between the set of natural transformations $F \to G$ and the set of functors $H : \mathsf{D} \to \mathsf{C}^{\to}$ such that

$$\begin{tikzcd} & \mathsf{C}^{\to} \arrow[d, "(S,T)"] \\ \mathsf{D} \arrow[ur, "H"] \arrow[r, "(F,G)"'] & \mathsf{C} \times \mathsf{C} \end{tikzcd}$$

commutes.

5. Let C and D be categories, where D is small. The *functor category* from D to C, denoted C^{D} or $\mathrm{Fun}(\mathsf{D}, \mathsf{C})$, has as objects the functors from D to C and as morphisms the natural transformations between them.

   (a) Explain how to define composition and identities in C^{D} so that it is actually a category and so that $\mathsf{C}^{\mathbf{2}}$ is isomorphic as a category to $\mathsf{C}^{\to}$, where $\mathbf{2}$ is the category with $\mathrm{Ob}\,\mathbf{2} = \{0, 1\}$ and only one non-identity morphism $t \in \mathbf{2}(0, 1)$.

   (b) Let $G : \mathsf{D} \to \mathsf{E}$ denote a functor between small categories. Show that "precomposition with G" defines a functor $G^* : \mathsf{C}^{\mathsf{E}} \to \mathsf{C}^{\mathsf{D}}$.

6. Why are the natural transformations $DU \to \mathrm{Id}_{\mathsf{Top}}$ and $\mathrm{Id}_{\mathsf{Top}} \to GU$ constructed in class not natural isomorphisms? (Note that if one of these natural transformations were a natural isomorphism, then the categories Top and Set would be equivalent, since $UD = UG = \mathrm{Id}_{\mathsf{Set}}$, so that the field of topology would be essentially the same as that of set theory!!)