Homotopical algebra Exercise Set 4

12.03.2018

Exercise 5 is to be handed in on 19.03.2018.

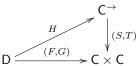
- 1. Let $F: \mathsf{C} \to \mathsf{D}$ be a functor. Show that F preserves isomorphisms, i.e., the image under F of an isomorphism in C is an isomorphism in D .
- 2. Let $F_{Ab}: \mathsf{Set} \to \mathsf{Ab}$ denote the free abelian group functor and $U: \mathsf{Ab} \to \mathsf{Set}$ the forgetful functor. Let $\tau: F_{Ab}U \to \mathrm{Id}_{\mathsf{Ab}}$ denote the natural transformation constructed in class.
 - (a) Show that there is a natural transformation $\iota: \mathrm{Id}_{\mathsf{Set}} \to UF_{\mathrm{Ab}}$ such that the composite

$$A \xrightarrow{\iota_A} UF_{Ab}(A) \xrightarrow{U(\tau_{(A,+,0)})} A$$

is the identity on the underlying set A of any abelian group (A, +, 0).

- (b) For any set X, show that $F_{Ab}(X)$ really is free on X, in the sense that for every abelian group (A, +, 0) and every function $f: X \to A$, there is a unique homomorphism $\hat{f}: F_{Ab}(X) \to (A, +, 0)$ such that $U(\hat{f}) \circ \iota_X = f$.
- 3. Let C be a category, and let $A \in \operatorname{Ob} C$. Let J be another category. Let $\Delta_A \colon \mathsf{J} \to \mathsf{C}$ denote the functor specified on objects by $\Delta_C(j) = A$ for all $j \in \operatorname{Ob} \mathsf{J}$ and on morphisms by $\Delta_C(f) = \operatorname{Id}_A$ for all $f \in \operatorname{Mor} \mathsf{C}$.
 - (a) Show that Δ_A is indeed a functor, which implies in particular that functors do not *reflect* isomorphisms, i.e., that the image of a morphism f under a functor is an isomorphism doesn't imply that f is an isomorphism.
 - (b) Let $F: J \to C$ be another functor. Show that a natural transformation $F \to \Delta_A$ can be interpreted as functor $J \to C/A$, while a natural transformation $\Delta_A \to F$ can be interpreted as a functor $J \to A/C$.

4. Let $F,G:\mathsf{D}\to\mathsf{C}$ be functors. Let $S,T:\mathsf{C}^\to\to\mathsf{C}$ denote the source and target functors seen in class. Show that there is a bijection between the set of natural transformations $F\to G$ and the set of functors $H:\mathsf{D}\to\mathsf{C}^\to$ such that



commutes.

- 5. Let C and D be categories, where D is small. The functor category from D to C, denoted C^D or $\operatorname{Fun}(D,C)$, has as objects the functors from D to C and as morphisms the natural transformations between them.
 - (a) Explain how to define composition and identities in C^D so that it is actually a category and so that C^2 is isomorphic as a category to C^{\to} , where $\mathbf{2}$ is the category with Ob $\mathbf{2} = \{0,1\}$ and only one non-identity morphism $t \in \mathbf{2}(0,1)$.
 - (b) Let $G: \mathsf{D} \to \mathsf{E}$ denote a functor between small categories. Show that "precomposition with G" defines a functor $G^*: \mathsf{C}^\mathsf{E} \to \mathsf{C}^\mathsf{D}$.
- 6. Why are the natural transformations $DU \to \operatorname{Id}_{\mathsf{Top}}$ and $\operatorname{Id}_{\mathsf{Top}} \to GU$ constructed in class not natural isomorphisms? (Note that if one of these natural transformations were a natural isomorphism, then the categories Top and Set would be equivalent, since $UD = UG = \operatorname{Id}_{\mathsf{Set}}$, so that the field of topology would be essentially the same as that of set theory!!)