Homotopical algebra
Exercise Set 3

05.03.2018

Exercise 5 is to be handed in on 12.03.2018.

1. Let C and D be categories. Explain how to define a category $\mathsf{C}\times\mathsf{D}$ such that $\mathrm{Ob}(\mathsf{C}\times\mathsf{D}) = \mathrm{Ob}\mathsf{C}\times\mathrm{Ob}\mathsf{D}$ and such that, in particular, $\mathsf{BG}\times\mathsf{BH} = \mathsf{B}(\mathsf{G}\times\mathsf{H})$ for all groups G and H.

2. Let C be a category, and let $A \in \mathrm{Ob}\mathsf{C}$. Motivated by the definition of $\mathsf{C}^{\rightarrow}$, explain how to define categories C/A such that $\mathrm{Ob}(\mathsf{C}/A) = \coprod_{B\in\mathrm{Ob}\mathsf{C}} \mathsf{C}(B, A)$ and A/C such that $\mathrm{Ob}(A/\mathsf{C}) = \coprod_{B\in\mathrm{Ob}\mathsf{C}} \mathsf{C}(A, B)$.

3. Let C be a category. An object A in C is *initial* if $\mathsf{C}(A, B)$ is a singleton for every $B \in \mathrm{Ob}\mathsf{C}$.

   (a) Prove that if A and A' are initial objects in C, then A and A' are isomorphic.

   (b) Determine whether each of the following categories has initial objects and, if so, describe them: Set, Top, Top_*, Gr, Ab, and BG (for some group G).

   (c) When does the arrow category $\mathsf{C}^{\rightarrow}$ have initial objects?

   We will usually use the notation $\emptyset$ to denote an initial object.

4. Let C be a category. An object A in C is *terminal* if $\mathsf{C}(B, A)$ is a singleton for every $B \in \mathrm{Ob}\mathsf{C}$.

   (a) Prove that if A and A' are terminal objects in C, then A and A' are isomorphic.

   (b) Determine whether each of the following categories has terminal objects and, if so, describe them: Set, Top, Top_*, Gr, Ab, and BG (for some group G).

   (c) When does the arrow category $\mathsf{C}^{\rightarrow}$ have terminal objects?

   We will usually use the notation e to denote a terminal object.

5. Let C be a category. Let S be a class of morphisms in C. Let $\mathrm{LLP}(S)$ and $\mathrm{RLP}(S)$ denote the classes of morphisms that have the *left lifting property* and *right lifting property*, respectively, with respect to S. In other words, $(i : A \to B) \in \mathrm{LLP}(S)$ if and only if for every $(s : C \to D) \in S$ and for every commuting diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "s"] \\ B \arrow[r, "g"] & D, \end{tikzcd}$$

there exists $\hat{g} \in \mathsf{C}(B, C)$ such that $s \circ \hat{g} = g$ and $\hat{g} \circ i = f$, and dually for $(p : A \to B) \in \mathrm{RLP}(S)$.

(a) Show that $\mathrm{LLP}(S)$ and $\mathrm{RLP}(S)$ are closed under composition.

(b) Show that a retract of a morphism in $\mathrm{LLP}(S)$ (respectively, $\mathrm{RLP}(S)$) is in $\mathrm{LLP}(S)$ (respectively, $\mathrm{RLP}(S)$) as well. (Retracts of morphisms are defined in arbitrary categories just as we defined them previously in Top.)

(c) Show that $\mathrm{LLP}(S)$ and $\mathrm{RLP}(S)$ contain all isomorphisms in C.

(d) When $\mathsf{C} = \mathsf{Set}$, and S is the class of all surjections, which functions belong to $\mathrm{LLP}(S)$? If S is the class of all injections, which functions belong to $\mathrm{RLP}(S)$?