Homotopical algebra
Exercise Set 12

14.05.2018

Exercise 3 is to be handed in on 28.05.2018.

1. Let $(\mathsf{M}, \mathsf{WE}_\mathsf{M}, \mathsf{Cof}_\mathsf{M}, \mathsf{Fib}_\mathsf{M})$ and $(\mathsf{N}, \mathsf{WE}_\mathsf{N}, \mathsf{Cof}_\mathsf{N}, \mathsf{Fib}_\mathsf{N})$ be model categories. Let $F : \mathsf{M} \to \mathsf{N}$ and $G : \mathsf{N} \to \mathsf{M}$ be functors.

   (a) Suppose that $\mathsf{WE}_\mathsf{N} = G^{-1}(\mathsf{WE}_\mathsf{M})$ and $\mathsf{Fib}_\mathsf{N} = G^{-1}(\mathsf{Fib}_\mathsf{M})$. Prove that $F \dashv G$ is a Quillen pair and that
   $$\mathrm{LLP}\big(G^{-1}(\mathsf{Fib}_\mathsf{M})\big) \subseteq \mathsf{WE}_\mathsf{N}.$$

   (b) Formulate and prove the dual result.

2. Let $(\mathsf{M}, \mathsf{WE}, \mathsf{Cof}, \mathsf{Fib})$ be a model category.

   (a) Let J denote any of the small categories of Exercise set 9, Exercise 3. With respect to the model structures defined there, prove that
   $$\begin{tikzcd} \mathsf{M}^\mathsf{J} \arrow[r, shift left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathsf{M} \arrow[l, shift left, "\Delta"] \end{tikzcd}$$
   is a Quillen pair.

   (b) Let J denote any of the small categories of Exercise set 10, Exercise 3. With respect to the model structures defined there, prove that
   $$\begin{tikzcd} \mathsf{M} \arrow[r, shift left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathsf{M}^\mathsf{J} \arrow[l, shift left, "\lim"] \end{tikzcd}$$
   is a Quillen pair.

3. Consider sSet equipped with the Kan-Quillen model structure, i.e., cofibrations are levelwise injections, while a weak equivalence is a simplicial

map the geometric realization of which is a weak homotopy equivalence. Let $K_\bullet$ be any simplicial set. Show that

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, shift left, "- \times K_\bullet"] \arrow[r, phantom, "\perp"] & \mathsf{sSet} \arrow[l, shift left, "\mathrm{Map}(K_\bullet, -)"] \end{tikzcd}$$

is a Quillen pair.

(**Hint:** Show that geometric realization preserves finite products, even though it's a left adjoint. Observe that, just as in the case of the fundamental group, $\pi_k\big(X \times Y, (x_0, y_0)\big) \cong \pi_k(X, x_0) \times \pi_k(Y, y_0)$ for every pair of pointed spaces (X, x_0), (Y, y_0).)

4. Consider a category M equipped with two model structures. What are the necessary conditions on these model structures to guarantee that $\mathrm{Id}_{\mathsf{M}} \dashv \mathrm{Id}_{\mathsf{M}}$ a Quillen pair, where M has one model structure on one side of the adjunction and the other structure on the other side? When is it a Quillen equivalence? Analyze and compare from this perspective the model structures you have studied on Set and the Hurewicz and Serre model structures on Top.