

A TORUS THEOREM FOR HOMOTOPY NILPOTENT GROUPS

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ABSTRACT. Nilpotency for discrete groups can be defined in terms of central extensions. In this paper, the analogous definition for spaces is stated in terms of principal fibrations having infinite loop spaces as fibers, yielding a new invariant we compare with classical cocategory, but also with the more recent notion of homotopy nilpotency introduced by Biedermann and Dwyer. This allows us to characterize finite homotopy nilpotent loop spaces in the spirit of Hubbuck's Torus Theorem, and corresponding results for p -compact groups and p -Noetherian groups.

INTRODUCTION

Biedermann and Dwyer, [6], used the stages of the Goodwillie tower of the identity to provide the first definition of homotopy nilpotency which interpolates between infinite loop spaces (homotopy abelian groups) and loop spaces (homotopy groups). Even though this might come as a surprise to those which are more accustomed to relate the Goodwillie tower of the identity to v_n -periodicity, as discovered by Arone and Mahowald, [3], there is a quite straightforward relationship between the essence of Goodwillie calculus, namely higher excision, and nilpotency.

In order to explain this let us first go back to group theoretical nilpotency, which can be defined in two equivalent ways, by using either commutators, or central extensions. The latter will be our point of view in this article, hence the nilpotency of a discrete group is understood as the minimal number of central extensions needed to construct the group. This group theoretical invariant has been successfully imported in homotopy theory by Berstein and Ganea, [4]. Their main motivation was to define a sensible notion of homotopy nilpotency for topological groups, such

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as compact Lie groups. Work of Hopkins, [19], and Rao, [30], gives a complete understanding of the classical Berstein–Ganea nilpotency of these objects. Only the torsion free ones, such as the 3-dimensional sphere \mathcal{S}^3 , an example which had been investigated by Porter in the early 60’s, [29], have finite nilpotency index.

However, by working in the homotopy category, significant information is lost. The natural analogue of nilpotency for loop spaces (i.e. group like spaces) is the following formal translation of central extensions. A homotopy abelian group should be an infinite loop space as it should be commutative not only up to homotopy, but up to all higher homotopies. Thus, we replace central extensions by principal fibrations whose fiber is an infinite loop space. The invariant which is introduced in this way is the minimal number of extensions by such principal fibrations needed to construct a given loop space. We call this invariant the *extension by principal fibrations length*, or *epfl* for short.

The relationship to Goodwillie calculus is provided by the structure of the Goodwillie tower. Let F be a functor from spaces to spaces and $F \rightarrow P_n F$ denote the n -excisive approximation of F . The homotopy fiber $D_n F$ of the natural transformation $P_n F \rightarrow P_{n-1} F$ is a homogeneous excisive functor and Goodwillie, [16], showed that they are classified by a spectrum together with an action of the symmetric group Σ_n . Concretely $D_n F(X)$ is the infinite loop space corresponding to the Σ_n -equivariant smash product of this spectrum with $X^{\wedge n}$. Even better, this fibration of functors is actually principal, [17].

In this paper, we prove the following.

Theorem 1.4 *Let Z be a pointed connected space. Then, we have the inequalities*

$$\mathrm{nil}_{\mathrm{BG}}(\Omega Z) \leq \mathrm{cocat}(Z) \leq \mathrm{epfl}(Z) \leq \mathrm{nil}_{\mathrm{BD}}(\Omega Z).$$

Here the two nilpotency invariants refer to the classical Berstein–Ganea nilpotency, $\mathrm{nil}_{\mathrm{BG}}$, and the Biedermann–Dwyer one. Even though at first sight the *epfl* invariant might seem somewhat naive, or maybe crude, it has two main advantages. First, it is actually very close to the Biedermann–Dwyer nilpotency and we do not know any example when they do not coincide; second, its simplicity enables us to gain a complete understanding of finite homotopy nilpotent loop spaces as we will see below.

If F is an n -excisive functor from the category of pointed spaces to pointed spaces and K a finite space, then $\Omega F(K)$ is a homotopy nilpotent group of class n , [6]. Thus Theorem 1.4 immediately implies that all $(n+1)$ -fold iterated Whitehead products vanish in $F(K)$. This has been proven originally in [10] relying on Goodwillie's generalized Blakers–Massey Theorem and the definition itself of an n -excisive functor. In [14] Eldred obtains another proof by analyzing Goodwillie's construction of the n -excisive approximation $P_n F$ of a functor F . She shows in fact that Whitehead products already vanish in $T_n F$, a functor directly related to Hopkins' symmetric cocategory, [20].

In the last part we focus on the Biedermann–Dwyer nilpotency of finite loop spaces and offer the following version of Hubbuck's Torus Theorem, [21], in this setting. The situation is very different with the new and stronger version of homotopy nilpotency than with the classical one.

Theorem 3.6 *Let (X, BX) be a finite and connected loop space. If $\mathrm{nil}_{\mathrm{BD}}(X)$ is finite, then X is a torus.*

There are analogous characterizations of homotopy nilpotency for p -compact groups and p -Noetherian groups. Two ingredients in the proof are the existence of a finite tower of principal fibrations and the effect of Neisendorfer's functor, [28], on the fibers of this tower.

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1. BACKGROUND AND FIRST RESULTS

In this section we briefly introduce the main ingredients in Theorem 1.4.

1.1. Nilpotency in the sense of Biedermann–Dwyer. We need some basic notions on algebraic theories to understand the nilpotency in the sense of Biedermann–Dwyer. They were introduced by Lawvere [23] to describe algebraic structures, and we refer the reader to [6, Section 2] for our interests. An algebraic theory is a small category T whose objects are indexed by the natural numbers $\{T_0, T_1, \dots, T_n, \dots\}$ and such that for $n \in \mathbb{N}$ the n -fold categorical coproduct of T_1 is naturally isomorphic to T_n . In our situation, the algebraic theory T will be required to be pointed and *simplicial*, which means that T is enriched over the

category \mathcal{S}_* of pointed simplicial sets. Thus we distinguish between *strict* and *homotopy* T -algebras, which are simplicial functors $X: T^{\text{op}} \rightarrow \mathcal{S}_*$ taking coproducts in T^{op} to products in \mathcal{S}_* strictly or up to homotopy, respectively.

Biedermann and Dwyer define homotopy nilpotent groups as homotopy \mathcal{G}_n -algebras in the category of pointed spaces, where \mathcal{G}_n is a simplicial algebraic theory constructed from the Goodwillie tower of the identity, [6, Definition 5.4]. Concretely, \mathcal{G}_n is the simplicial category which has for each natural number $k \geq 0$ exactly one object given by $\mathcal{G}_n(k) = \prod_k \Omega(P_n(\text{id}))^{\text{inj}}$. The functor $P_n(\text{id})$ lives in the category of functors from finite pointed spaces to pointed spaces, it is the n -excisive approximation of the identity functor, and $(P_n(\text{id}))^{\text{inj}}$ denotes the fibrant replacement in the injective model structure [5]. The simplicial set of morphisms $\mathcal{G}_n(k, l)$ is the space of natural transformations $\prod_k \Omega(P_n(\text{id}))^{\text{inj}} \rightarrow \prod_l \Omega(P_n(\text{id}))^{\text{inj}}$.

Hence, an *homotopy nilpotent group* X of *class* $\leq n$ is the value at 1 of a simplicial functor \tilde{X} from \mathcal{G}_n to pointed spaces, which commutes up to homotopy with products. In fact, the homotopy nilpotent group is the whole functor \tilde{X} and we isolate abusively the space $\tilde{X}(1)$. This space comes in particular with a loop space structure since there is a structure map $\tilde{X}(1) \times \tilde{X}(1) \simeq \tilde{X}(2) \rightarrow \tilde{X}(1)$ corresponding to the loop space product.

In this paper we write $\text{nil}_{\text{BD}}(X) \leq n$ and, we say that X is a *homotopy nilpotent group* if it is of class $\leq n$ for some n . The two extremes of this theory are well understood: loop spaces and infinite loop spaces can be described as homotopy algebras over the theories \mathcal{G}_∞ and \mathcal{G}_1 respectively. This invariant

1.2. Nilpotency in the sense of Berstein–Ganea. Applying the functor π_0 to the simplicial algebraic theory \mathcal{G}_n gives us the ordinary theory of n -nilpotent groups, Nil_n , [6, Theorem 8.1]. Now, product preserving functors to the homotopy category of pointed spaces, $N: \text{Nil}_n^{\text{op}} \rightarrow \text{Ho}(\text{Spaces}_*)$, or in other words, Nil_n -algebras in the homotopy category of pointed spaces, are exactly the n -nilpotent groups in the sense of Berstein–Ganea [10, Proposition 4.2]. Recall that for a loop space ΩZ , the nilpotency of ΩZ in the sense of Berstein–Ganea is the least integer n for which the $(n+1)$ -st commutator map $\varphi_{n+1}: (\Omega Z)^{n+1} \rightarrow \Omega Z$ is homotopically trivial [4, Definition 1.7]. In this paper we write then $\text{nil}_{\text{BG}}(\Omega Z) \leq n$. If such an integer n exists then all the $(n+1)$ -fold iterated Whitehead products vanish in Z , [4, Theorem 4.6].

The following example shows that nilpotency in the sense of Berstein–Ganea does not capture the subtlety of the loop space structure.

Example 1.1. Following the lines of [27, Section 5], let $f : \mathbb{C}P^\infty \rightarrow K(\mathbb{Z}, 4)$ represent the square of the fundamental class of the infinite complex projective space. Define $Z = \text{Fib} f$ and $X = \Omega Z$. As a space X splits as a product $S^1 \times \mathbb{C}P^\infty$ but as a loop it doesn't and X is not an infinite loop space, meaning that its Berstein–Ganea nilpotency is 1 but its Biedermann–Dwyer nilpotency isn't. It is thus important to remember the loop space structure, here and in the calculation of the Biedermann–Dwyer nilpotency.

The next two invariants are gathered in the next paragraph since they are both related to fibrations, and in some sense, the second one is a refinement of the first one.

1.3. Inductive cocategory and epfl. Different notions of cocategory are available on the market, see for example [18], [20], and [27]. In this paper we concentrate on the inductive cocategory introduced by Ganea, [15], concretely, its normalized version. Thus, for a connected pointed space Z , $\text{cocat}(Z) = 0$ if and only if Z is contractible and, for any $n \geq 1$, $\text{cocat}(Z) \leq n$ if there exists a fibration $F \rightarrow Y \rightarrow B$ with $\text{cocat}(Y) \leq n - 1$ and F dominates Z (equivalently Z is a homotopy retract of F). It is clear from the definition that $\text{cocat}(Z) \leq 1$ if and only if Z is dominated by a loop space.

The relation between the cocategory and the nilpotency in the sense of Berstein–Ganea is given by the inequality $\text{nil}_{\text{BG}}(\Omega Z) \leq \text{cocat}(Z)$, [15, Theorem 2.12].

A variant of the previous definition is the following.

Definition 1.2. We say that a connected pointed Z is an extension by principal fibrations of length 0 if and only if Z is a contractible space and, for $n \geq 1$, a space Z is an *extension by principal fibrations of length $\leq n$* , if there exists a tower of principal fibrations

$$\begin{array}{ccccccc} Z \simeq Z_n & \longrightarrow & Z_{n-1} & \longrightarrow & \dots & \longrightarrow & Z_1 & \longrightarrow & \star \\ \uparrow & & \uparrow & & & & \parallel & & \\ F_n & & F_{n-1} & & & & F_1 & & \end{array}$$

where all fibers $F_k = \text{Fib}(Z_k \rightarrow Z_{k-1})$ are infinite loop spaces. In that case, we write $\text{epfl}(Z) \leq n$.

Lemma 1.3. *Let Z be a pointed connected space. Then $\text{cocat}(Z) \leq \text{epfl}(Z)$.*

Proof. If $\text{epfl}(Z) \leq n$, there exists by definition a tower of principal fibrations $Z \rightarrow Z_{n-1} \rightarrow \cdots \rightarrow Z_1 \rightarrow *$. Thus the fibration $Z \rightarrow Z_{n-1}$ is classified by a map $Z_{n-1} \rightarrow BF_n$, where $F_n = \text{Fib}(Z \rightarrow Z_{n-1})$. Hence, $\text{cocat}(Z) \leq \text{cocat}(Z_{n-1}) + 1$ and, by induction, we see that $\text{cocat}(Z) \leq \text{cocat}(\star) + n = n$. \square

We are now ready to prove the main theorem in this section. We exploit a characterization of homotopy nilpotent groups through excisive functors. For if F is an n -excisive functor (so F sends strongly homotopy co-Cartesian $(n+1)$ -cubes to homotopy Cartesian ones) then, for every finite space K , $\Omega F(K)$ is a homotopy nilpotent group [6, Theorem 9.2]. Even better, Biedermann and Dwyer show that every functor \tilde{X} associated to a homotopy nilpotent group of class $\leq n$ is of the form ΩF where F is n -excisive, [7].

We prove the following:

Theorem 1.4. *Let Z be a pointed connected space. Then, we have the inequalities*

$$\text{nil}_{\text{BG}}(\Omega Z) \leq \text{cocat}(Z) \leq \text{epfl}(Z) \leq \text{nil}_{\text{BD}}(\Omega Z).$$

Proof. In view of the previous paragraphs, nothing needs to be done for the first two inequalities. The first one is [15, Theorem 2.12] and the second one is Lemma 1.3. Suppose that ΩZ is a homotopy nilpotent group of class $\leq n$. Then, by [7], there exists an n -excisive functor F and a space K such that ΩZ and $\Omega F(K)$ are weakly equivalent as loop spaces. Therefore we have an equivalence $Z \simeq F(K)$. Now, the Goodwillie tower for $F \simeq P_n F$ yields a tower

$$F(K) \simeq P_n F(K) \rightarrow P_{n-1} F(K) \rightarrow \cdots \rightarrow P_1 F(K) \rightarrow \star$$

whose fibers $D_k F(K)$ are infinite loop spaces, and $P_k F(K) \rightarrow P_{k-1} F(K)$ are actually a principal fibrations classified by $P_{k-1} F(K) \rightarrow BD_k F(K)$, [17, Lemma 2.2]. This directly implies that $\text{epfl}(Z) \leq n$. \square

When ΩZ is not only a loop space, but an infinite loop space, all inequalities are indeed equalities. This comes from the fact that homotopy abelian groups in the sense of Biedermann and Dwyer are infinite loop spaces.

Proposition 1.5. *Let X be an infinite loop space. Then $\text{nil}_{\text{BG}}(X) = \text{nil}_{\text{BD}}(X) = 1$.* \square

This approach leads us to an alternate and more direct proof of a result obtained by a direct calculation in [10] (see also Eldred's point of view, [14]).

Corollary 1.6. *Let F be any n -excisive functor from the category of pointed spaces to pointed spaces. Then all $(n+1)$ -fold iterated Whitehead products vanish in $F(K)$ for every finite space K .*

Proof. Since F is an n -excisive functor, $\Omega F(K)$ is a homotopy nilpotent group of class $\leq n$, [6, Theorem 9.2]. Thus $\text{nil}_{\text{BG}}(\Omega F(K)) \leq n$ by Theorem 1.4 and all the $(n+1)$ -fold iterated Whitehead products vanish in $F(K)$ (see §1.2). \square

This application highlights the simplicity of the arguments when using the relationship between the classical notions of nilpotency, nil_{BG} and cocat , and the more recent ones, epfl and nil_{BD} .

Remark 1.7. The upper bound we obtain above is not the best possible. Arone, Dwyer, and Lesh show for example that a $(2n-1)$ -excisive functor for which $P_{n-1}F \simeq \star$ takes its values in infinite loop spaces, [2, Theorem 4.2]. In principle, the extension by homogeneous layers is n -fold, but this result shows that all Whitehead products vanish in such a space.

We end this section by showing that all nilpotency notions coincide for discrete groups. In order to do so, we use the relation of \mathcal{G}_n with the set-valued theory of ordinary n -nilpotent groups Nil_n :

Theorem 1.8. *Let G be a discrete nilpotent group. Then*

$$\text{nil}_{\text{BG}}(G) = \text{cocat}(BG) = \text{epfl}(BG) = \text{nil}_{\text{BD}}(G).$$

Proof. Let G be a nilpotent group of class n , i.e. $n = \text{nil}_{\text{BG}}(G)$. There is then a set valued Nil_n -algebra given by the functor $\underline{G}: \text{Nil}_n \rightarrow \text{Sets}_*$ where $\underline{G}(k) = G^k$.

Considering the isomorphism of categories $\text{Nil}_n \cong \pi_0 \mathcal{G}_n$ [6, Theorem 8.1] and viewing pointed sets as a full subcategory of \mathcal{S}_* , shows that G is a homotopy nilpotent group of class $\leq n$ since the functor $\underline{G} \circ \pi_0: \mathcal{G}_n \rightarrow \mathcal{S}_*$ extends to a simplicial functor as all mapping spaces between discrete groups are discrete. It is therefore an homotopy \mathcal{G}_n -algebra and

$$n = \text{nil}_{\text{BG}}(G) \leq \text{cocat}(BG) \leq \text{epfl}(BG) \leq \text{nil}_{\text{BD}}(G) \leq n.$$

\square

We would be remiss in not saying a word about examples of spaces for which these inequalities are sharp. All of our attempts have been unsuccessful to find an example of a space X for which $\text{epfl}(X)$ is strictly less than $\text{nil}_{\text{BD}}(\Omega X)$.

2. TOWERS OF PRINCIPAL FIBRATIONS AND NEISENDORFER' TYPE FUNCTOR

The aim of this section is to show that the effect of certain homotopical localization functors on loop spaces with finite epfl is predictable, see Definition 1.2. We start with the description of the localization functor we have in mind.

Let $f : B\mathbf{Z}/p \rightarrow \star$ and g be a universal $H\mathbf{F}_p$ -equivalence (meaning that L_g , localization at g , is mod p homological localization). We set $L = L_{f \vee g}$, the Bousfield localization functor which “kills” $B\mathbf{Z}/p$ and “inverts” all mod p homology equivalences.

Remark 2.1. Quite often, the effect of L on a space Y is that of Neisendorfer's functor $(P_{B\mathbf{Z}/p}(-))_p^\wedge$, nullification followed by p -completion. It is so for example when Y is a connected infinite loop space with p -torsion fundamental group, since McGibbon proved in this case that $(P_{B\mathbf{Z}/p}Y)_p^\wedge$ is contractible. In general however one has to iterate nullification and mod p homological localization. The first, mild, issue with Neisendorfer's functor is that p -completion does not coincide with homological localization in general, and the second one is that even on p -good spaces it is not an idempotent functor, since the $B\mathbf{Z}/p$ -nullification of a p -complete space does not remain p -complete in general. The point of using L instead of Neisendorfer's functor is that L is an idempotent functor, and therefore all the machinery in [11] can be applied. Indeed, the proofs of the results in this section frequently use properties of fiberwise localization (see for example Dror Farjoun's [11, Section 1.F]).

We first show that the effect of L on loop spaces can be read in the universal cover and then, that considering the universal cover does not increase the length of the extension by principal fibrations of a space.

Lemma 2.2. *Let X be a connected loop space such that $\pi_1 X$ is p -torsion, and let $X\langle 1 \rangle$ be the universal cover of X . Then $L(X)$ is contractible if $L(X\langle 1 \rangle)$ is so.*

Proof. Let $P = \pi_1 X$. We have a fibration sequence $X\langle 1 \rangle \rightarrow X \rightarrow BP$. The universal cover $X\langle 1 \rangle$ being L -acyclic, fiberwise localization shows that $X \rightarrow BP$ in an L -equivalence, i.e. $LX \simeq LBP$. The latter is contractible because we assume P is a p -group, hence $P_{B\mathbf{Z}/p}BP \simeq \star$. \square

Lemma 2.3. *Let $f : Z \rightarrow Y$ be a map of connected spaces whose homotopy fiber is an infinite loop space. Then the homotopy fiber of the induced map on universal*

covers $f\langle 1\rangle: Z\langle 1\rangle \rightarrow Y\langle 1\rangle$ is an infinite loop space as well. Moreover, if f is a principal fibration, then so is $f\langle 1\rangle$.

Proof. Let F be the homotopy fiber of f , a non-connected space in general. The homotopy fiber of $f\langle 1\rangle: Z\langle 1\rangle \rightarrow Y\langle 1\rangle$ is a covering of the base point component of F . Its fundamental group is the subgroup of $\pi_1 F$ corresponding to the quotient $\text{Ker}(\pi_1 Z \rightarrow \pi_1 Y)$. It is thus an infinite loop space as well. The second statement can be deduced for example from the general work in [12]. \square

This readily implies that if $\text{epfl}(Z) \leq n$, then $\text{epfl}(Z\langle 1\rangle) \leq n$. We refine this elementary observation to obtain a version where the homotopy fibers in the tower are simply connected, a key technical fact for the sequel.

Lemma 2.4. *Let $f: Z \rightarrow Y$ be a principal fibration of simply connected spaces whose homotopy fiber F is a (connected) infinite loop space. There exists then a factorization $f: Z \rightarrow \bar{Y} \rightarrow Y$ such that the homotopy fiber $\text{Fib}(Z \rightarrow \bar{Y})$ is the universal cover $F\langle 1\rangle$.*

Proof. The classifying space BF of the connected infinite loop space F is simply connected and we define $A = \pi_2 BF$. Using the second Postnikov fibration sequence

$$B(F\langle 1\rangle) \rightarrow BF \rightarrow K(A, 2)$$

we construct \bar{Y} as the homotopy fiber of the composite map

$$Y \rightarrow BF \rightarrow K(A, 2).$$

The desired factorization exists by construction and the homotopy fiber of the induced map $Z \rightarrow \bar{Y}$ is the simply connected infinite loop space $F\langle 1\rangle$. \square

The key technical result to be able to understand the effect of Neisendorfer's type functor on a space with finite epfl is the following. We need to modify the tower of principal fibrations for the universal cover so as to obtain simply connected loop spaces. It might be possible to do so without losing the principality but we state here a weaker result.

Proposition 2.5. *Let Z be a connected space with $\text{epfl}(Z) \leq n$. Then the universal cover $Z\langle 1\rangle$ is also an extension by (not necessarily principal) fibrations of length n , where the fibers are simply connected infinite loop spaces.*

Proof. Since $\text{epfl}(Z) \leq n$ there exists a tower of principal fibrations

$$Z \simeq Z_n \longrightarrow Z_{n-1} \longrightarrow \dots \longrightarrow Z_1 \longrightarrow \star$$

such that the homotopy fibers $\text{Fib}(Z_k \rightarrow Z_{k-1})$ are infinite loop spaces. Since Z is connected the spaces Z_k can be chosen to be connected as well. Lemma 2.3 implies that the tower of universal covers $Y_k = Z_k\langle 1 \rangle$ is made of principal fibrations that has also infinite loop spaces as homotopy fibers.

We are thus left with a tower $Z\langle 1 \rangle \simeq Y_n \rightarrow Y_{n-1} \rightarrow \dots \rightarrow Y_1 \rightarrow \star$ of simply connected spaces. The fibers G_k however are not simply connected in general, but only connected. We modify thus the spaces Y_k for $k < n$ to fix this. Since $G_1 = Y_1$ is simply connected, we can assume by induction that Y_{n-1} is an extension by fibrations of length $n - 1$, with simply connected infinite loop spaces as fibers.

We use now the factorization $Y_n \rightarrow \bar{Y}_{n-1} \rightarrow Y_{n-1}$ of Lemma 2.4. Since the homotopy fiber of the first map is a simply connected infinite loop space, we must show to conclude that \bar{Y}_{n-1} is an extension by fibrations of length $n - 1$ with simply connected infinite loop spaces as fibers. Consider now the homotopy fiber H of the composite map $\bar{Y}_{n-1} \rightarrow Y_{n-1} \rightarrow Y_{n-2}$. The homotopy fiber H fits by construction into a fibration sequence

$$H \rightarrow G_{n-1} \rightarrow K(A_n, 2)$$

where $A_n = \pi_2 B G_n$. However, since G_{n-1} is simply connected by induction hypothesis, the map $G_{n-1} \rightarrow K(A_n, 2)$ factors through the second Postnikov section of G_{n-1} . This is a map of infinite loop spaces, which implies that H is an infinite loop space. Therefore, the tower $\bar{Y}_{n-1} \rightarrow Y_{n-2} \rightarrow \dots \rightarrow Y_1$ exhibits \bar{Y}_{n-1} as an extension by fibrations of length $n - 1$ with *connected* infinite loop spaces as fibers. The induction hypothesis allows us to conclude that it is also an extension by principal fibrations of length $n - 1$ with *simply connected* infinite loop spaces as fibers. \square

We finally describe the effect of L on loops on spaces of finite extension by principal fibrations:

Theorem 2.6. *Let ΩZ be a connected space with p -torsion fundamental group. If $\text{epfl}(Z)$ is finite, then $L(\Omega Z)$ is contractible.*

Proof. By Lemma 2.2 it is enough to prove that $L((\Omega Z)\langle 1 \rangle)$ is contractible. Let $n = \text{epfl}(Z)$. Then $\text{epfl}(\Omega Z) \leq n$ and by Proposition 2.5 there exists a tower of

fibrations

$$(1) \quad (\Omega Z)\langle 1 \rangle \simeq Y_n \rightarrow Y_{n-1} \rightarrow \cdots \rightarrow Y_1 \rightarrow \star$$

whose fibers are simply connected infinite loop spaces F_k . By McGibbon’s result [25] $LF_k \simeq \star$ for any $n \geq k \geq 1$. Applying fiberwise localization allows us then to conclude that (1) is a tower of L -equivalences, and therefore $L((\Omega Z)\langle 1 \rangle) \simeq \star$. \square

Example 2.7. Consider a torus $(S^1)^m$. This is an infinite loop space, a.k.a. a homotopy abelian group (nilpotent of class one). The fundamental group is not p -torsion and the fact that $L((S^1)^m) \simeq K(\oplus \mathbf{Z}_p^\wedge, 1)$ does not contradict the previous theorem.

3. HOMOTOPY NILPOTENCY OF p -COMPLETE LOOP SPACES

Hubbuck’s Torus Theorem, [21], characterizes classical homotopy commutative finite H -spaces as tori, i.e. finite products of circles. Mod p versions were established shortly after by Hubbuck and Aguadé–Smith [1], and many extensions of this theorem have been obtained by relaxing the finiteness conditions. Moving from mod p finite H -spaces to those having a finitely generated cohomology algebra over the Steenrod algebra yields a characterization of homotopy commutative H -spaces at the prime 2 and loop spaces on H -spaces at odd primes as products of a torus and a Pottsikov piece, [8, Corollary 7.4]. We shift gears in this section and consider homotopy commutative loop spaces in the Biedermann–Dwyer sense, and more generally homotopy nilpotent groups with finiteness conditions.

Theorem 3.1. *Let (X, BX) be a connected p -compact group. Then, the loop space X is homotopy nilpotent if and only if X is a p -complete torus.*

Proof. By applying the cellularization functor CW_{S^2} to X , we obtain the universal cover of X , $CW_{S^2}(X) = X\langle 1 \rangle \simeq \text{Fib}(X \rightarrow B\pi_1(X))$. Cellularization is a continuous functor that preserves products up to homotopy [11, Theorem 2.E.10], therefore $X\langle 1 \rangle$ is a homotopy nilpotent group if X is so.

Let us assume that $X\langle 1 \rangle$ is a homotopy nilpotent group. Since $X\langle 1 \rangle$ is a connected loop space with p -torsion fundamental group, Theorem 2.6 tells us that $L(X\langle 1 \rangle)$ must be contractible. However, by Miller’s proof of the Sullivan conjecture [26], we have that $P_{B\mathbf{Z}/p}X\langle 1 \rangle \simeq X\langle 1 \rangle$, so $LX\langle 1 \rangle \simeq X\langle 1 \rangle$. Therefore $X\langle 1 \rangle$ is contractible so that X is homotopy equivalent to $B\pi_1(X)$. Now, by [13, Remark 2.2], the fundamental group of a p -compact group is a finite direct sum of

copies of cyclic groups \mathbb{Z}/p^r and copies of p -adic integers \mathbb{Z}_p^\wedge . Since X is \mathbb{F}_p -finite, this implies that X must be a p -complete torus. \square

Corollary 3.2. *Let G be a simply connected compact Lie group. Then $\text{nil}_{\text{BD}}(G_p^\wedge)$ is infinite.* \square

This implies that the nilpotency in the sense of Berstein–Ganea, and the nilpotency in the sense of Biedermann–Dwyer do not coincide in general.

Example 3.3. McGibbon’s p -local examples of classically homotopy commutative compact Lie groups can be translated in the p -complete setting, [24]. The group $Sp(2)$ at the prime 3 and the exceptional group G_2 at the prime 5 have Berstein–Ganea nilpotency 1, whereas we have just seen that they have infinite Biedermann–Dwyer nilpotency. In [22] Kaji and Kishimoto study examples of p -compact groups that are nilpotent in the sense of Berstein–Ganea but they are not p -completed tori, so by Theorem 3.1 they are not homotopy nilpotent. In other words, there exist many loop spaces X for which $\text{nil}_{\text{BG}}(X) < \infty$, but $\text{nil}_{\text{BD}}(X) = \infty$. This is the case for example for E_8 at the prime 41 where $\text{nil}_{\text{BG}}((E_8)_{41}^\wedge) = 3$, [22, Theorem 1.6].

We move now from p -compact groups to p -Noetherian groups. Let us recall from [9] that a p -Noetherian group is a loop space (X, BX) where BX is p -complete and $H^*(X; \mathbb{F}_p)$ is a finitely generated (Noetherian) \mathbb{F}_p -algebra.

Theorem 3.4. *Let (X, BX) be a connected p -Noetherian group. If X is a homotopy nilpotent group, then BX is a 2-stage Postnikov piece fitting in a fibration sequence $K(\mathbb{Z}_p^\wedge, 3)^r \rightarrow BX \rightarrow K(P, 2) \times ((BS^1)_p^\wedge)^s$ where P is a finite abelian p -group.*

Proof. In [9], for a p -Noetherian group X , the authors construct a fibration

$$K(P, 2)_p^\wedge \rightarrow BX \rightarrow BY = (P_{\Sigma B\mathbb{Z}_p} BX)_p^\wedge$$

where P is a finite direct sum of copies of cyclic groups and Prüfer groups, and Y is a p -compact group. Since we have a series of homotopy equivalences

$$Y \simeq \Omega BY \simeq \Omega(P_{\Sigma B\mathbb{Z}_p} BX)_p^\wedge \simeq (\Omega(P_{\Sigma B\mathbb{Z}_p} BX))_p^\wedge$$

and $(\Omega(P_{\Sigma B\mathbb{Z}_p} BX))_p^\wedge$ is homotopy equivalent to $(P_{B\mathbb{Z}_p} \Omega BX)_p^\wedge$ [11, Theorem 3.A.1], we get that $Y \simeq (P_{B\mathbb{Z}_p} X)_p^\wedge$. Also observe that all the homotopy equivalences are loop maps [11, Lemma 3.A.3]. This allows us to say that (Y, BY) is obtained in a functorial way from (X, BX) using the nullification and the p -completion functors.

Both functors are continuous and preserve products up to homotopy. Therefore, if X is homotopy nilpotent, so is Y .

Now, by Theorem 3.1, since Y is a p -compact group, it must be a p -complete torus. The result now follows by taking in account that for Prüfer groups, p -completion shifts dimension by one: $K(\mathbb{Z}/p^\infty, 2)_p^\wedge \simeq K(\mathbb{Z}_p^\wedge, 3)$. \square

Corollary 3.5. *Let (X, BX) be a connected p -Noetherian group and assume that X a homotopy nilpotent group. Then X is a homotopy 2-nilpotent group.*

Proof. We know by the previous theorem that the homotopy groups $\pi_s(BX)$ vanish unless $2 \leq s \leq 3$. The non-trivial homotopy groups of the simply connected space BX live thus in the metastable range, therefore by [6, Example 9.8], $\Omega BX \simeq X$ is a homotopy 2-nilpotent group. \square

We finally come back to the *integral* Torus Theorem, [21]. The classical statement is that, if $\text{nil}_{\text{BG}}(X) = 1$, then X has the homotopy type of a torus.

Theorem 3.6. *Let (X, BX) be a finite and connected loop space. If $\text{nil}_{\text{BD}}(X)$ is finite, then X is a torus.*

Proof. For finite and connected loop spaces (X, BX) , the p -completion $(X_p^\wedge, BX_p^\wedge)$ is a p -compact group for every prime p , which by Theorem 3.1, means that X_p^\wedge is a p -complete torus for every prime. The arithmetic square shows then that X itself is a torus. \square

Following the ideas of Rector [31], it is commonly accepted that compact Lie groups should be thought of as finite loop spaces, and that the structural data of the Lie group have to be read by means of homotopy invariants. Within this framework, connected non abelian compact Lie groups are expected to be highly non nilpotent. The previous result shows that nilpotency in the sense of Biederman–Dwyer is the right notion in contrast with classical nilpotency in the sense of Bernstein–Ganea. So for example, by a classical result of Porter [29] the sphere \mathcal{S}^3 is nilpotent in the sense of Bernstein–Ganea, $\text{nil}_{\text{BG}}(\mathcal{S}^3) = 3$, but since \mathcal{S}^3 is not a torus, $\text{nil}_{\text{BD}}(\mathcal{S}^3) = \infty$.

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