

## Homological Localizations Preserve 1-Connectivity

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**ABSTRACT.** Every generalized homology theory  $E$  yields a localization functor  $\mathbf{L}_E$  that sends the  $E$ -equivalences to homotopy equivalences. We prove that if  $X$  is any 1-connected space, then  $\mathbf{L}_E X$  is also 1-connected, for every generalized homology theory  $E$ . This is deduced from a result by Hopkins and Smith stating that if  $K(\mathbb{Z}, 2)$  is  $E$ -acyclic then  $E$  is trivial.

### Introduction

A number of results in the literature suggest that idempotent functors in the homotopy category of spaces preserve 1-connectivity, although no proof of this fact has so far been given. One of the earliest examples is localization with respect to ordinary homology, which in fact preserves  $n$ -connectivity for all  $n$ ; see [1].

In the same article [1], Bousfield proved the existence of localization with respect to any generalized homology theory  $E$ ; that is, a functor  $\mathbf{L}_E$  which assigns to every space  $X$  a space  $\mathbf{L}_E X$  together with a natural map  $X \rightarrow \mathbf{L}_E X$  which is terminal in the homotopy category among  $E$ -equivalences with source  $X$ . (An  $E$ -equivalence is a map  $X \rightarrow Y$  inducing isomorphisms  $E_n(X) \cong E_n(Y)$  for all  $n$ .)

In [9], Mislin showed that  $K$ -theory localization does not preserve  $n$ -connectivity in general, since for example  $\pi_3(\mathbf{L}_K S^{2p+2}; \mathbb{Z}/p) \neq 0$  for every odd prime  $p$ . However, Mislin also proved in [9] that the  $K$ -localization of every 1-connected space is 1-connected. Further evidence of the fact that 1-connectivity could be preserved by arbitrary idempotent functors in the homotopy category was given by Neisendorfer in [10] and by Tai in his detailed study of the problem in [11].

It is therefore natural to address the question of whether or not localizations with respect to generalized homology theories preserve 1-connectivity. Such localizations were thoroughly discussed by Bousfield in [2], where a description was given of their effect on abelian Eilenberg–Mac Lane spaces. The main tool was an arithmetic square, already exploited by Mislin in [9], allowing one to determine the  $E$ -localization of a space (with some restrictions on the fundamental group) from its  $E\mathbb{Z}/p$ -localizations and rational coherence data.

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Our main result is that  $\mathbf{L}_E X$  is 1-connected if  $X$  is 1-connected, for any generalized homology theory  $E$ . This follows by combining the methods of Bousfield in [2] with a result proved by Hopkins and Smith in [8], according to which a  $K(\mathbb{Z}, 2)$  is never  $E$ -acyclic if  $E$  is nontrivial. We note, however, that  $K(\mathbb{Z}, 3)$  is  $K\mathbb{Z}/p$ -acyclic for all  $p$ , by [9, Corollary 2.3]. It is known that, if  $L$  is any homotopy idempotent functor, then  $LK(\mathbb{Z}, n)$  is necessarily a  $K(A, n)$  where  $A$  is either zero or a commutative ring with 1, for all  $n$ ; see [5]. If  $L = \mathbf{L}_E$  for some nontrivial homology theory  $E$ , then the possibility that  $A = 0$  has been discarded for  $n = 2$  in [8], and this opens the way to substantial improvements of earlier results or to new results as in this article.

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### 1. Torsion homology theories

Throughout the paper we denote by  $E$  a spectrum or the associated homology theory. For an abelian group  $R$ , the corresponding spectrum with coefficients in  $R$  is defined as  $ER = E \wedge SR$  where  $SR$  is the Moore spectrum of type  $(R, 0)$ . The only cases of interest in this article are  $R = \mathbb{Z}/p$  and  $R$  a subring of  $\mathbb{Q}$ . A spectrum  $E$  is called *torsion* if  $E\mathbb{Q}$  is contractible. The ordinary Eilenberg–Mac Lane spectrum with coefficients in  $R$  is denoted by  $HR$ . We denote by  $\mathbb{Z}_p^\wedge$  the  $p$ -adics, by  $\mathbb{Z}(p^\infty)$  the Prüfer group  $\bigcup_{n=1}^\infty \mathbb{Z}/p^n$  and, for a set of primes  $P$ , we denote by  $\mathbb{Z}_P$  the integers localized at  $P$ .

In this first section we concentrate on mod  $p$  homology theories, where  $p$  is any prime. Using the Atiyah–Hirzebruch spectral sequence, one sees that if  $E$  is any homology theory, then every  $H\mathbb{Z}/p$ -equivalence is an  $E\mathbb{Z}/p$ -equivalence; details are given in [9, § 1]. Hence, all  $E\mathbb{Z}/p$ -local spaces are  $H\mathbb{Z}/p$ -local and there is a natural transformation of functors  $\mu: \mathbf{L}_{H\mathbb{Z}/p} \rightarrow \mathbf{L}_{E\mathbb{Z}/p}$ .

We next prove that, if  $X$  is connected, then the induced homomorphism

$$\mu_*: \pi_1(\mathbf{L}_{H\mathbb{Z}/p} X) \rightarrow \pi_1(\mathbf{L}_{E\mathbb{Z}/p} X)$$

is surjective. This result is essentially contained in the proof of Proposition 7.1 in [2], as we next recall for the sake of completeness. The argument is based on Bousfield’s version of the Whitehead theorem (cf. [2, Theorem 5.2]), stating that if  $R$  is  $\mathbb{Z}/p$  or a subring of  $\mathbb{Q}$ , and  $f: X \rightarrow Y$  is a map inducing isomorphisms  $H_i(X; R) \cong H_i(Y; R)$  for  $i < n$  and an epimorphism  $H_n(X; R) \twoheadrightarrow H_n(Y; R)$ , where  $n \geq 1$ , then  $f$  also induces isomorphisms  $\pi_i(\mathbf{L}_{HR} X) \cong \pi_i(\mathbf{L}_{HR} Y)$  for  $i < n$  and an epimorphism  $\pi_n(\mathbf{L}_{HR} X) \twoheadrightarrow \pi_n(\mathbf{L}_{HR} Y)$ .

**THEOREM 1.1.** *Let  $E$  be any homology theory and  $p$  any prime. Then, for every connected space  $X$ , the natural homomorphism  $\mu_*: \pi_1(\mathbf{L}_{H\mathbb{Z}/p} X) \rightarrow \pi_1(\mathbf{L}_{E\mathbb{Z}/p} X)$  is surjective.*

**PROOF.** The claim is obvious if  $E\mathbb{Z}/p$  is trivial. If  $E\mathbb{Z}/p$  is not trivial, then  $K(\mathbb{Z}/p, 1)$  is not  $E\mathbb{Z}/p$ -acyclic, as shown in [2, Proposition 2.2]. Since the natural map  $\mu: \mathbf{L}_{H\mathbb{Z}/p} X \rightarrow \mathbf{L}_{E\mathbb{Z}/p} X$  is an  $E\mathbb{Z}/p$ -equivalence, we obtain an isomorphism

$$(1.1) \quad \mu_*: H_1(\mathbf{L}_{H\mathbb{Z}/p} X; \mathbb{Z}/p) \cong H_1(\mathbf{L}_{E\mathbb{Z}/p} X; \mathbb{Z}/p)$$

using [2, Proposition 2.1] or [5, Theorem 1.3], according to which  $K(\mathbb{Z}/p, 1)$  is  $E\mathbb{Z}/p$ -local. By the generalized Whitehead theorem stated above,  $\mu$  induces then an epimorphism  $\pi_1(\mathbf{L}_{H\mathbb{Z}/p}X) \twoheadrightarrow \pi_1(\mathbf{L}_{E\mathbb{Z}/p}X)$ , since  $\mathbf{L}_{E\mathbb{Z}/p}X$  is  $H\mathbb{Z}/p$ -local.  $\square$

**COROLLARY 1.2.** *If  $E$  is any torsion homology theory and  $X$  is 1-connected, then  $\mathbf{L}_E X$  is also 1-connected.*

**PROOF.** As in [2], we denote by  $\mathcal{P}E$  the set of primes  $p$  such that  $\pi_*(E)$  is not uniquely  $p$ -divisible. By [2, Proposition 7.1], for each torsion homology theory  $E$  and every 1-connected space  $X$ , we have a homotopy equivalence

$$\mathbf{L}_E X \simeq \prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p} X.$$

Now recall from [1] that  $\mathbf{L}_{H\mathbb{Z}/p} X$  is 1-connected if  $X$  is 1-connected. Therefore, Theorem 1.1 tells us that  $\mathbf{L}_E X$  is 1-connected.  $\square$

Before discussing non-torsion homology theories, we need to study the second homotopy group  $\pi_2(\mathbf{L}_{E\mathbb{Z}/p} X)$  when  $X$  is 1-connected. The following result is the main input in our discussion.

**THEOREM 1.3.** *Let  $E$  be a homology theory and  $p$  any prime. Suppose that  $E\mathbb{Z}/p$  is nontrivial. Then either  $K(\mathbb{Z}/p, 2)$  or  $K(\mathbb{Z}_p^\wedge, 2)$  is  $E\mathbb{Z}/p$ -local.*

**PROOF.** The classification of acyclicity patterns for Eilenberg–Mac Lane spaces given by Bousfield in [2, § 4] implies that  $\mathbf{L}_{E\mathbb{Z}/p} K(\mathbb{Z}, n) = K(A, n)$  for each  $n \geq 1$ , where the group  $A$  can be  $\mathbb{Z}_p^\wedge$ , or  $\mathbb{Z}/p^i$  for some  $i \geq 1$ , or zero. In [8], it is shown that if a reduced homology theory vanishes on  $K(\mathbb{Z}, 2)$ , then it is trivial. (Thus, nontrivial mod  $p$  homology theories of type IV-1 as defined in [2, § 4] do not exist.) Therefore, if  $\mathbf{L}_{E\mathbb{Z}/p}$  is nontrivial, then the localization  $\mathbf{L}_{E\mathbb{Z}/p} K(\mathbb{Z}, 2)$  is necessarily  $K(\mathbb{Z}_p^\wedge, 2)$  or  $K(\mathbb{Z}/p^i, 2)$  for some  $i \geq 1$ . In the latter case,  $K(\mathbb{Z}/p, 2)$  cannot be  $E\mathbb{Z}/p$ -acyclic, as one sees by induction using the fibre sequences

$$K(\mathbb{Z}/p, 2) \rightarrow K(\mathbb{Z}/p^i, 2) \rightarrow K(\mathbb{Z}/p^{i-1}, 2).$$

Hence,  $K(\mathbb{Z}/p, 2)$  is  $E\mathbb{Z}/p$ -local, by [2, Proposition 2.1] or [5, Lemma 1.4].  $\square$

If  $K(\mathbb{Z}/p, 2)$  is  $E\mathbb{Z}/p$ -local and  $X$  is 1-connected, then, using the fact that  $\mu: \mathbf{L}_{H\mathbb{Z}/p} X \rightarrow \mathbf{L}_{E\mathbb{Z}/p} X$  is an  $E\mathbb{Z}/p$ -equivalence, we obtain as in (1.1) an isomorphism

$$(1.2) \quad \mu_*: H_2(\mathbf{L}_{H\mathbb{Z}/p} X; \mathbb{Z}/p) \cong H_2(\mathbf{L}_{E\mathbb{Z}/p} X; \mathbb{Z}/p).$$

Thus, the homomorphism  $\pi_2(\mathbf{L}_{H\mathbb{Z}/p} X) \rightarrow \pi_2(\mathbf{L}_{E\mathbb{Z}/p} X)$  induced by  $\mu$  is surjective, by the generalized Whitehead theorem.

Now suppose that  $K(\mathbb{Z}_p^\wedge, 2)$  is  $E\mathbb{Z}/p$ -local and  $X$  is 1-connected. Similarly as in the previous case, since  $\mu$  is an  $E\mathbb{Z}/p$ -equivalence, we have an isomorphism

$$(1.3) \quad \mu^*: \text{Hom}(\pi_2(\mathbf{L}_{E\mathbb{Z}/p} X), \mathbb{Z}_p^\wedge) \cong \text{Hom}(\pi_2(\mathbf{L}_{H\mathbb{Z}/p} X), \mathbb{Z}_p^\wedge).$$

In order to use this information, we recall the following concept from [4, VI.3] and [7]. An abelian group  $A$  is called *Ext- $p$ -complete* if the natural homomorphism  $A \rightarrow \text{Ext}(\mathbb{Z}(p^\infty), A)$  derived from the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[1/p] \rightarrow \mathbb{Z}(p^\infty) \rightarrow 0$$

is an isomorphism. Equivalently, an abelian group  $A$  is *Ext- $p$ -complete* if and only if both  $\text{Hom}(\mathbb{Z}[1/p], A) = 0$  and  $\text{Ext}(\mathbb{Z}[1/p], A) = 0$ . As explained in [4, VI.4],

Ext- $p$ -complete abelian groups are uniquely  $q$ -divisible for primes  $q \neq p$ , and they admit a canonical  $\mathbb{Z}_p^\wedge$ -module structure.

An Ext- $p$ -complete abelian group  $A$  is called *adjusted* if the quotient  $A/TA$  of  $A$  by its torsion subgroup  $TA$  is  $p$ -divisible (hence divisible). Thus,  $A$  is adjusted if and only if  $A$  does not admit any torsion-free Ext- $p$ -complete quotients other than zero. Since  $TA \otimes \mathbb{Z}(p^\infty) = 0$ , it also follows that an Ext- $p$ -complete abelian group  $A$  is adjusted if and only if  $A \otimes \mathbb{Z}(p^\infty) = 0$ .

**THEOREM 1.4.** *Let  $E$  be a homology theory and  $p$  a prime. Suppose that  $E\mathbb{Z}/p$  is nontrivial. Then, for every 1-connected space  $X$ , the cokernel of the natural homomorphism  $\mu_*: \pi_2(\mathbf{L}_{H\mathbb{Z}/p}X) \rightarrow \pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$  is an adjusted Ext- $p$ -complete abelian group, which is zero if  $K(\mathbb{Z}/p, 2)$  is  $E\mathbb{Z}/p$ -local.*

**PROOF.** The spaces  $\mathbf{L}_{H\mathbb{Z}/p}X$  and  $\mathbf{L}_{E\mathbb{Z}/p}X$  are  $H\mathbb{Z}/p$ -local. The abelian groups  $\pi_2(\mathbf{L}_{H\mathbb{Z}/p}X)$  and  $\pi_2(\mathbf{L}_{E\mathbb{Z}/p}X)$  are thus Ext- $p$ -complete, by [1, Theorem 5.5]. Hence,  $\text{Coker } \mu_*$  is Ext- $p$ -complete, since the cokernel of any homomorphism between Ext- $p$ -complete abelian groups is Ext- $p$ -complete. If  $K(\mathbb{Z}/p, 2)$  is  $E\mathbb{Z}/p$ -local, then we already proved, by means of (1.2), that  $\text{Coker } \mu_*$  is zero. Thus, we assume that  $K(\mathbb{Z}_p^\wedge, 2)$  is  $E\mathbb{Z}/p$ -local. In this case, the isomorphism displayed in (1.3) shows that  $\text{Hom}(\text{Coker } \mu_*, \mathbb{Z}_p^\wedge) = 0$ . For an abelian group  $A$ , if  $\text{Hom}(A, \mathbb{Z}_p^\wedge) = 0$  then we have  $\text{Hom}(A \otimes \mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) = 0$  by adjunction. Since  $A \otimes \mathbb{Z}(p^\infty)$  is a  $p$ -torsion divisible abelian group, we may infer that  $A \otimes \mathbb{Z}(p^\infty) = 0$  and this implies that  $A/TA$  is  $p$ -divisible, as we needed. (In fact, an Ext- $p$ -complete abelian group  $A$  is adjusted if and only if the condition  $\text{Hom}(A, \mathbb{Z}_p^\wedge) = 0$  holds. This has also been pointed out in [3, Lemma 7.7].)  $\square$

## 2. Non-torsion homology theories

In this section we deal with non-torsion homology theories. In this case, there is an arithmetic square allowing one to compute  $E$ -localizations of 1-connected spaces by combining mod  $p$  data and rational data. Specifically, the following diagram is a homotopy pull-back square if  $X$  is 1-connected (and also under less restrictive conditions; see [2, Proposition 7.2]). Recall that  $\mathcal{P}E$  denotes the set of primes  $p$  such that  $\pi_*(E)$  is not uniquely  $p$ -divisible.

$$\begin{array}{ccc} \mathbf{L}_E X & \longrightarrow & \prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p} X \\ \downarrow & & \downarrow \\ \mathbf{L}_{H\mathbb{Q}} X & \longrightarrow & \mathbf{L}_{H\mathbb{Q}} \left( \prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p} X \right). \end{array}$$

We also need the following remark.

**LEMMA 2.1.** *Suppose given a set of primes  $P$  and an adjusted Ext- $p$ -complete abelian group  $A_p$  for all  $p \in P$ . The rationalization  $\prod_{p \in P} A_p \rightarrow \left( \prod_{p \in P} A_p \right) \otimes \mathbb{Q}$  is then an epimorphism.*

PROOF. Fix any prime  $q \in P$ . Then we have  $A_q \otimes \mathbb{Z}(q^\infty) = 0$  since  $A_q$  is adjusted, and  $(\prod_{p \neq q} A_p) \otimes \mathbb{Z}(q^\infty) = 0$  as well, since  $\prod_{p \neq q} A_p$  is uniquely  $q$ -divisible. Therefore,  $(\prod_{p \in P} A_p) \otimes \mathbb{Z}(q^\infty) = 0$ . This shows that  $(\prod_{p \in P} A_p) \otimes \mathbb{Q}/\mathbb{Z} = 0$ , which proves our claim.  $\square$

Our main result is the following.

**THEOREM 2.2.** *Let  $E$  be any homology theory and let  $X$  be 1-connected. Then  $\mathbf{L}_E X$  is also 1-connected.*

PROOF. By Corollary 1.2, we may assume that  $E$  is not torsion. Our strategy is to compare the arithmetic squares for  $E$  and ordinary homology  $H\mathbb{Z}_{\mathcal{P}E}$ . The natural maps  $\mu: \mathbf{L}_{H\mathbb{Z}/p} X \rightarrow \mathbf{L}_{E\mathbb{Z}/p} X$  yield a commutative diagram

$$\begin{array}{ccccc}
 Y & \longrightarrow & F' & \longrightarrow & F \\
 \downarrow & & \downarrow & & \downarrow \\
 F'' & \longrightarrow & \prod_{p \in \mathcal{P}E} \mathbf{L}_{H\mathbb{Z}/p} X & \xrightarrow{\mu} & \prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p} X \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{L}_{H\mathbb{Q}} F'' & \longrightarrow & \mathbf{L}_{H\mathbb{Q}} \left( \prod_{p \in \mathcal{P}E} \mathbf{L}_{H\mathbb{Z}/p} X \right) & \longrightarrow & \mathbf{L}_{H\mathbb{Q}} \left( \prod_{p \in \mathcal{P}E} \mathbf{L}_{E\mathbb{Z}/p} X \right)
 \end{array}$$

where each row and each column is a fibre sequence. The four spaces in the lower right square are 1-connected by Corollary 1.2. Therefore, all the fibres except perhaps  $Y$  are connected. The group  $\pi_1(F'')$  is the product of the cokernels of the homomorphisms  $\mu_*: \pi_2(\mathbf{L}_{H\mathbb{Z}/p} X) \rightarrow \pi_2(\mathbf{L}_{E\mathbb{Z}/p} X)$ , so it is a product of adjusted Ext- $p$ -complete groups, by Theorem 1.4. Hence, Lemma 2.1 tells us that the induced homomorphism  $\pi_1(F'') \rightarrow \pi_1(\mathbf{L}_{H\mathbb{Q}} F'')$  is surjective. This implies that  $Y$  is connected as well, so the homomorphism  $\pi_1(F') \rightarrow \pi_1(F)$  is surjective.

From the arithmetic square for  $E$  we see that  $\mathbf{L}_E X$  is 1-connected if and only if the boundary homomorphism  $\pi_2(\mathbf{L}_{H\mathbb{Q}} X) \rightarrow \pi_1(F)$  is surjective. Consider now the fibre sequence  $F' \rightarrow \mathbf{L}_{H\mathbb{Z}_{\mathcal{P}E}} X \rightarrow \mathbf{L}_{H\mathbb{Q}} X$  appearing in the arithmetic square for  $H\mathbb{Z}_{\mathcal{P}E}$ . Since we know that  $\mathbf{L}_{H\mathbb{Z}_{\mathcal{P}E}} X$  is 1-connected, the homomorphism  $\pi_2(\mathbf{L}_{H\mathbb{Q}} X) \rightarrow \pi_1(F')$  is surjective. The composite  $\pi_2(\mathbf{L}_{H\mathbb{Q}} X) \rightarrow \pi_1(F') \rightarrow \pi_1(F)$  is thus also surjective, as we needed.  $\square$

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