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Relating Postnikov pieces with the Krull filtration: a spin-off of Serre's theorem

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Abstract. We characterize H-spaces which are p-torsion Postnikov pieces of finite type by a cohomological property together with a necessary acyclicity condition. When the mod p cohomology of an H-space is finitely generated as an algebra over the Steenrod algebra we prove that its homotopy groups behave like those of a finite complex. In particular, a p-complete infinite loop space has a finite number of non-trivial homotopy groups if and only if its mod p cohomology satisfies this finiteness condition.

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Introduction

When does cohomological information allow to determine whether or not a given space is a Postnikov piece? In the 50's Serre showed that a non-trivial 1-connected finite complex cannot be a Postnikov piece. He also proved that the same happens for CW-complexes with finite mod 2 cohomology [16], and predicted the same behavior at odd primes. After Miller's solution to Sullivan's conjecture [12], this was proved for spaces with finite mod p cohomology by McGibbon and Neisendorfer in [11].

The discovery of Lannes' T-functor enabled to extend this result to a larger family of spaces. Indeed, Lannes and Schwartz proved in [9] that non-trivial 1-connected spaces with locally finite mod p cohomology, as unstable module over the Steenrod algebra, cannot be Postnikov pieces. Finally, in [6], Dwyer and Wilkerson showed that it is also true for 2-connected spaces for which the module of indecomposable elements in the mod p cohomology is locally finite, including in particular the case where the cohomology is finitely generated as an algebra.

Observe that the locally finite unstable modules form the 0th stage of the Krull-Schwartz filtration $\{\mathscr{U}_n\}$ of the category $\mathscr{U}$ of unstable modules over the Steenrod algebra $\mathscr{A}_p$, a filtration introduced in relation with Kuhn's realizability conjectures,

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see [8] and [15]. Schwartz characterizes the unstable modules M lying in $\mathcal{U}_n$ as those for which $\overline{T}^{n+1}M = 0$, see [14, Theorem 6.2.4]. We will use the standard notation T_V for Lannes' T functor, where V is an elementary abelian group, and simply T when $V = \mathbb{Z}/p$. Finally $\overline{T}$ denotes the reduced version of T.

Thus, the Dwyer-Wilkerson result deals with 2-connected spaces X such that the module of indecomposable elements $QH^*(X; \mathbb{F}_p)$ is in $\mathcal{U}_0$. In this context we obtain the following extension for H-spaces.

Theorem 1.2. *Let $n \geq 0$ and X be an $(n+2)$-connected H-space such that $T_V H^*(X; \mathbb{F}_p)$ is of finite type for any elementary abelian p-group V. Assume that $QH^*(X; \mathbb{F}_p)$ lies in $\mathcal{U}_n$. If X is not contractible, then it has infinitely many non-trivial homotopy groups with p-torsion and the iterated loop space $\Omega^{n+1}X$ has infinitely many non-trivial k-invariants.*

Serre's result and its generalizations state conditions on the mod p cohomology implying that a space *is not* a Postnikov piece. Our next objective is to give conditions to ensure that a space is a Postnikov piece (p-torsion since mod p cohomology does not detect q-primary information for primes $q \neq p$).

It is well-known that p-torsion Eilenberg-Mac Lane spaces are $B\mathbb{Z}/p$-acyclic, that is, their $B\mathbb{Z}/p$-nullification is contractible (we refer the reader to the book [7] for details about nullification). This implies that p-torsion Postnikov pieces are $B\mathbb{Z}/p$-acyclic as well, so that a first test to find out if a p-torsion space is a Postnikov piece would be to apply the nullification functor $P_{B\mathbb{Z}/p}$. However, this is not a sufficient condition as illustrated by the obvious example of $\prod_{n\geq 1} K(\mathbb{Z}/p, n)$. When dealing with H-spaces, we offer a necessary and sufficient condition in terms of cohomology and nullification.

Theorem 2.2. *An H-space X is a p-torsion Postnikov piece of finite type if and only if $P_{B\mathbb{Z}/p}X$ is contractible and $H^*(X; \mathbb{F}_p)$ is a finitely generated algebra over $\mathcal{A}_p$.*

Other examples of H-spaces with finitely generated cohomology as an algebra over the Steenrod algebra are the highly connected covers of finite H-spaces. It follows from Neisendorfer's theorem [13] that their $B\mathbb{Z}/p$-nullification is not contractible. We prove in Proposition 3.1 that, under this finiteness condition, there are basically no other H-spaces with infinitely many non-trivial homotopy groups than the highly connected covers of mod p finite H-spaces.

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1 The homotopy groups of H-spaces

The original theorem [6, Theorem 1.3] by Dwyer and Wilkerson about the homotopy groups of certain 2-connected spaces relies on the equivalence between a cohomological condition and a topological one. Namely, the loop space of a p-complete space is $B\mathbb{Z}/p$-null if and only if the module of indecomposable elements in $H^*(X;\mathbb{F}_p)$ is locally finite, [6, Proposition 3.2]. In fact, this result can be understood as a reduction step to the theorem of Lannes and Schwartz: If X is 2-connected and $QH^*(X;\mathbb{F}_p)$ is locally finite, then ΩX is $B\mathbb{Z}/p$-null as we just recalled; thus the cohomology $H^*(\Omega X;\mathbb{F}_p)$ is locally finite, which implies by [9] that ΩX has infinitely many non-trivial homotopy groups (unless it is contractible).

When X is an H-space, we were able to obtain an extension of [6, Proposition 3.2] using the Krull filtration of the category of unstable modules over $\mathscr{A}_p$.

Theorem 1.1 [4, Theorem 5.3]. *Let X be a connected H-space such that $T_V H^*(X;\mathbb{F}_p)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X;\mathbb{F}_p) \in \mathscr{U}_n$ if and only if $\Omega^{n+1}X$ is a $B\mathbb{Z}/p$-null space.* □

We obtain then, as in [6], a result on the homotopy groups of sufficiently connected spaces satisfying the conditions of our theorem.

Theorem 1.2. *Let $n \geq 0$ and X be an $(n+2)$-connected H-space such that $T_V H^*(X;\mathbb{F}_p)$ is of finite type for any elementary abelian p-group V. Assume that $QH^*(X;\mathbb{F}_p)$ lies in $\mathscr{U}_n$. If X is not contractible, then it has infinitely many non-trivial homotopy groups with p-torsion and the iterated loop space $\Omega^{n+1}X$ has infinitely many non-trivial k-invariants.*

Proof. By Theorem 1.1, $\Omega^{n+1}X$ is a $B\mathbb{Z}/p$-null space. We know thus from [4, Theorem 5.5] (compare with [1, Theorem 7.2]) that the homotopy fiber F of the nullification map $X \to P_{B\mathbb{Z}/p}X$ is a p-torsion Postnikov piece with its homotopy groups concentrated in degrees from 1 to $n+1$.

We infer from the homotopy long exact sequence that $P_{B\mathbb{Z}/p}X$ is simply connected and not contractible, because X is $(n+2)$-connected and not contractible. Therefore, the Lannes-Schwartz theorem [9] applies. The space $P_{B\mathbb{Z}/p}X$ must have an infinite number of non-trivial homotopy groups with p-torsion, and so does X.

The assertion about the k-invariants follows from the fact that an Eilenberg-Mac Lane space $K(A,m)$ is not $B\mathbb{Z}/p$-null if A contains p-torsion. □

Corollary 1.3. *Let $n \geq 0$ and X be a p-complete H-space such that $T_V H^*(X;\mathbb{F}_p)$ is of finite type, $H^*(X;\mathbb{F}_p)$ is $(n+2)$-connected, and $QH^*(X;\mathbb{F}_p) \in \mathscr{U}_n$. Then X is the $(n+2)$-connected cover of a $B\mathbb{Z}/p$-null H-space.*

Proof. As X is p-complete, the "Connectivity Lemma" [2, I.6.1] implies that X itself is $(n+2)$-connected. The fibration $F \to X \to P_{B\mathbb{Z}/p}X$ used in the proof of Theorem 1.2 exhibits now X as a highly connected cover of a $B\mathbb{Z}/p$-null space. □

2 On H-spaces that are Postnikov pieces

Whereas algebraic conditions implying that a space is not a Postnikov piece are frequently encountered in the literature, a characterization of Postnikov pieces in terms of their cohomology seems out of reach. Our aim in this section is to provide a satisfactory answer for H-spaces. Let us first look at the cohomology of an H-space with finitely many p-torsion homotopy groups.

Proposition 2.1. *Let X be an H-space which is a p-torsion Postnikov piece of finite type. Then $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$.*

Proof. The homotopy group of a p-torsion Eilenberg-Mac Lane space of finite type is a finite direct sum of cyclic groups $\mathbb{Z}/p^n$ and Prüfer groups $\mathbb{Z}_{p^\infty}$. The cohomology of such spaces has been computed by Cartan and Serre. It is finitely generated as an algebra over $\mathscr{A}_p$ (see for example [14, Section 8.4]).

In [4, Theorem 6.1] we proved that the cohomology of the total space of an H-fibration over $K(A,n)$ is a finitely generated algebra over $\mathscr{A}_p$, if so are the cohomology of both the fiber and the base. Therefore the result follows by induction on the number of homotopy groups of the H-space X. □

We offer now our characterization by combining Proposition 2.1 with a result analogous to [13, Lemma 2.1] on the $B\mathbb{Z}/p$-nullification of p-torsion Postnikov pieces.

Theorem 2.2. *An H-space X is a p-torsion Postnikov piece of finite type if and only if $P_{B\mathbb{Z}/p}X$ is contractible and $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$.*

Proof. If $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$, then by [4, Lemma 7.1] the module $QH^*(X;\mathbb{F}_p)$ belongs to $\mathscr{U}_n$ for some n and $T_V H^*(X;\mathbb{F}_p)$ is of finite type for any V. Therefore, Theorem 1.1 applies and $\Omega^{n+1}X$ is $B\mathbb{Z}/p$-null. Now, Bousfield's description [1, Theorem 7.2] of the homotopy fiber of the nullification map $X \to P_{B\mathbb{Z}/p}X$ (see [4, Theorem 5.5] in this concrete setting) tells us that it is a p-torsion Postnikov piece. As $P_{B\mathbb{Z}/p}X$ is contractible, X itself is a Postnikov piece, and since $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$, an elementary Serre spectral sequence argument shows that X is of finite type.

Conversely, if X is a p-torsion Postnikov piece which is an H-space, then its $B\mathbb{Z}/p$-nullification is contractible. This statement follows from the fact that p-torsion Eilenberg-Mac Lane spaces are $B\mathbb{Z}/p$-acyclic and $P_{B\mathbb{Z}/p}$ preserves fibrations in which the fiber is $B\mathbb{Z}/p$-acyclic ([7, Theorem 1.H.1]). We conclude by Proposition 2.1. □

Remark 2.3. When X is not an H-space, this characterization fails. Consider for example X, the homotopy fiber of the nullification map $BS^3 \to P_{B\mathbb{Z}/p}BS^3 \simeq \mathbb{Z}[1/p]_\infty BS^3$ (see [5, Theorem 1.7, Lemma 6.2]). Then $P_{B\mathbb{Z}/p}X$ is contractible by [7, Theorem 1.H.2], and $H^*(X;\mathbb{F}_p)$ is isomorphic to $H^*(BS^3;\mathbb{F}_p)$, hence finitely generated as an algebra. Notwithstanding X is not a p-torsion Postnikov piece.

We wish to mention that there is an obvious way to apply our results to spaces that are not H-spaces, namely by considering their loop space.

Corollary 2.4. *A 1-connected space X is a p-torsion Postnikov piece of finite type if and only if $P_{\Sigma B\mathbb{Z}/p}X$ is contractible and $H^*(\Omega X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

Proof. A space is a Postnikov piece if and only if its loop space is so. Thus Theorem 2.2 applies to the connected H-space ΩX and we conclude since $P_{B\mathbb{Z}/p}\Omega X \simeq \Omega P_{\Sigma B\mathbb{Z}/p}X$, [7, Theorem 3.A.1]. □

It would be nice to find a characterization in terms of the cohomology of X itself rather than the mod p loop space cohomology.

3 Connected covers of finite H-spaces

This section is devoted to analyze the nature of H-spaces whose mod p cohomology is a finitely generated algebra over $\mathcal{A}_p$ but that are not Postnikov pieces. Examples of such H-spaces are the highly connected covers of simply connected mod p finite H-spaces (such as odd dimensional spheres completed at odd primes). Such spaces have obviously infinitely many non-trivial homotopy groups, as a direct consequence of Serre's original theorem [16, Théorème 10] and its generalization given by McGibbon and Neisendorfer [11, Theorem 1]. In this framework we prove that there are basically no other H-spaces which do have infinitely many non-trivial homotopy groups: Any H-space with finitely generated mod p cohomology as an algebra over the Steenrod algebra differs from a mod p finite one by only a finite number of homotopy groups. In other words, some iterated loop space of such an H-space coincides with the iterated loop space of a mod p finite H-space.

Proposition 3.1. *Let X be an H-space such that $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathcal{A}_p$. Then there exist an integer n and an H-space Y with finite* mod p *cohomology such that the $(n+2)$-connected covers of Y and X are homotopy equivalent. Moreover, when $P_{B\mathbb{Z}/p}X$ is not contractible, X has infinitely many non-trivial homotopy groups.*

Proof. The space Y is obtained as the $B\mathbb{Z}/p$-nullification of X. Its mod p cohomology is finite because it is both finitely generated as an algebra over $\mathcal{A}_p$ and locally finite as an unstable module, see [4, Theorem 7.2]. The integer n is the smallest one such that $QH^*(X;\mathbb{F}_p)$ belongs to $\mathcal{U}_n$, which exists since $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathcal{A}_p$. Moreover X and Y differ by a finite number of homotopy groups (concentrated in degrees from 1 to $n+1$) because the homotopy fiber of $X \to Y$ is a p-torsion Postnikov piece, see [4, Theorem 5.5]. □

This yields, together with Corollary 1.3, a criterion to recognize cohomologically the $(n+2)$-connected cover of a mod p finite H-space. When $n=0$, this does not bring anything new since the universal cover of a mod p finite H-space is again a mod p finite H-space, which is even 2-connected [3, Theorem 6.10].

Corollary 3.2. *Let $n \geq 0$ and X be a p-complete connected H-space such that $H^*(X;\mathbb{F}_p)$ is an $(n+2)$-connected finitely generated $\mathcal{A}_p$-algebra such that $QH^*(X;\mathbb{F}_p)$ is in $\mathcal{U}_n$. Then X is the $(n+2)$-connected cover of an H-space with finite* mod p *cohomology.* □

4 A variation with Neisendorfer's functor

In Section 2 we considered the nullification functor $P_{B\mathbb{Z}/p}$. Next we explain how to obtain analogous results for the functor $(P_{B\mathbb{Z}/p}(-))^\wedge_p$ introduced by Neisendorfer in [13]. However we need to add an extra condition on the fundamental group (for instance, S^1 is a $B\mathbb{Z}/p$-null space which is a Postnikov piece).

Proposition 4.1. *Let X be a p-complete H-space with finite fundamental group. Then X is a Postnikov piece of finite type if and only if $(P_{B\mathbb{Z}/p}X)^\wedge_p$ is contractible and $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$.*

Proof. If $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$, consider the fibration $F \to X \to P_{B\mathbb{Z}/p}X$, in which the fiber F is a p-torsion Postnikov piece. Since H-fibrations are preserved by p-completion and $(P_{B\mathbb{Z}/p}X)^\wedge_p$ is contractible, we see that $X^\wedge_p$ is homotopy equivalent to the p-completion of a p-torsion Postnikov piece.

Conversely, if X is a connected p-complete Postnikov piece of finite type with finite fundamental group, then $(P_{B\mathbb{Z}/p}X)^\wedge_p$ is contractible by Neisendorfer's result [13, Lemma 2.1] and the cohomology is finitely generated as an algebra over $\mathscr{A}_p$ by the same argument as in Proposition 2.1. □

The connectivity assumption in Theorem 1.2 cannot be relaxed because of the obvious example of $K(\mathbb{Z}, n+2)$. In fact this is essentially the unique $(n+1)$-connected H-space which is a Postnikov piece such that $QH^*(X;\mathbb{F}_p)$ lies in $\mathscr{U}_n$.

Proposition 4.2. *Let X be an $(n+1)$-connected H-space for some integer $n \geq 0$ such that $T_V H^*(X;\mathbb{F}_p)$ is of finite type for any elementary abelian p-group V. Assume that the module of indecomposable elements $QH^*(X;\mathbb{F}_p)$ lies in $\mathscr{U}_n$ and that X is a Postnikov piece. Then X is, up to p-completion, homotopy equivalent to the product of finitely many copies of $K(\mathbb{Z}^\wedge_p, n+2)$.*

Proof. By Proposition 4.1 $(P_{B\mathbb{Z}/p}X)^\wedge_p$ is contractible, so X itself is, up to p-completion, homotopy equivalent to the fiber F of the nullification map $X \to P_{B\mathbb{Z}/p}X$. Since $\Omega^{n+1}X$ is a $B\mathbb{Z}/p$-null space, we know that the homotopy groups of F are concentrated in degrees from 1 to $n+1$. The connectivity assumption implies that $F^\wedge_p$ must be $(n+1)$-connected. Thus the only non-trivial homotopy group of F is $\pi_{n+1}F$ and it must be a finite product of copies of $\mathbb{Z}_{p^\infty}$, since $K(\mathbb{Z}_{p^\infty}, n+1)^\wedge_p \simeq K(\mathbb{Z}^\wedge_p, n+2)$. □

Finally we propose a characterization of Postnikov pieces which are infinite loop spaces.

Proposition 4.3. *Let X be a p-complete infinite loop space with finite fundamental group. Then X is a Postnikov piece of finite type if and only if $H^*(X;\mathbb{F}_p)$ is a finitely generated algebra over $\mathscr{A}_p$.*

Proof. The $B\mathbb{Z}/p$-nullification of a connected infinite loop space with p-torsion fundamental group is trivial up to p-completion by McGibbon's main theorem in [10]. We conclude by Proposition 4.1. □

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