

ON THE HOMOTOPY GROUPS OF p -COMPLETED CLASSIFYING SPACES

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ABSTRACT. Among the generalizations of Serre’s theorem on the homotopy groups of a finite complex we isolate the one proposed by Dwyer and Wilkerson. Even though the spaces they consider must be 2-connected, we show that it can be used to both recover known results and obtain new theorems about p -completed classifying spaces.

1. INTRODUCTION

In his celebrated paper in 1953 [22] Serre proved that a simply connected finite CW -complex has infinitely many non-trivial homotopy groups. He conjectured that it should actually have infinitely many non-trivial homotopy groups with 2-torsion. This was proved by McGibbon and Neisendorfer in 1983 [18] by using Miller’s solution [19] of the Sullivan conjecture. They show it happens for any simply connected CW -complex with finite mod 2 cohomology, replacing thereby the geometric finiteness condition by a purely algebraic one. Later, in 1986, Lannes and Schwartz [14] were able to relax the finiteness condition to *locally finite* mod p cohomology, i.e. the cohomology is a direct limit of finite unstable modules over the Steenrod algebra. So they proved Serre’s conjecture for “Miller spaces”, that is, 1-connected spaces X for which the space of pointed maps from $B\mathbb{Z}/p$ to X is contractible.

In 1990, Dwyer and Wilkerson [11] show the following generalization of Serre’s conjecture. Let X be a 2-connected CW -complex of finite type with non trivial mod p cohomology and such that the module of indecomposable elements in $H^*(X; \mathbb{F}_p)$ is locally finite. Then infinitely many homotopy groups of X contain p -torsion. This *new* algebraic condition obviously includes the previous ones, namely spaces with

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finite or locally finite mod p cohomology. Moreover their condition enables us to study spaces with finitely generated mod p cohomology, because the indecomposable module is then finite.

The cost in the Dwyer-Wilkerson theorem is that one has to work with 2-connected spaces. As they say, “the example of $\mathbb{C}P^\infty$ shows that it would not be enough to assume that X is 1-connected”. We prove that this is basically the only simply connected Postnikov piece with locally finite module of indecomposables.

Theorem 3.3 *Let X be a p -complete CW-complex of finite type such that the module $QH^*(X; \mathbb{F}_p)$ of indecomposable elements is locally finite. Then one of the following properties is satisfied:*

- (1) X is aspherical,
- (2) $X\langle 1 \rangle$ is a $K(\mathbb{Z}_p^\wedge, 2)^n$,
- (3) X has infinitely many homotopy groups with p -torsion.

In the last case moreover the space ΩX has infinitely many non-trivial k -invariants.

In particular we show that the Lannes-Schwartz result can be seen as a consequence, see Corollary 3.4. In fact the above result can even be applied to understand spaces having a non-trivial fundamental group, such as classifying spaces of discrete groups. A very exciting problem in homotopy theory is to determine the behavior of the p -completion of such BG 's. When G is a finite group Levi proves in [15] that either $(BG)_p^\wedge$ is again an Eilenberg-Mac Lane space or it has infinitely many non-trivial homotopy groups. Later Bastardas and Descheemaker discovered the same phenomenon holds for any virtually nilpotent group. This is done in [2] for torsion free groups and the general case is solved in [1]. We show that all these results can be deduced from our version of Dwyer-Wilkerson theorem and we obtain the same statement for certain quasi p -perfect groups with finite virtual mod p cohomology and also for the new concept of p -local finite group, due to Broto, Levi, and Oliver [8].

Theorem 5.1 *Let X be the classifying space of a member of the following four families:*

- (1) *finite groups,*
- (2) *p -local finite groups,*

- (3) *finitely generated virtually nilpotent groups,*
- (4) *quasi p -perfect groups of finite virtual mod p cohomology.*

Then the p -completion of X is either aspherical or it has infinitely many homotopy groups with p -torsion. In this case moreover the space $\Omega(X)_p^\wedge$ has infinitely many non-trivial k -invariants.

2. LOCAL LOOP SPACES

The theorem on which the whole paper is built relies on the equivalence between the algebraic condition that the module of indecomposables $QH^*(X; \mathbb{F}_p)$ be locally finite and the topological one that the loop space ΩX is $B\mathbb{Z}/p$ -local. The proof of [11, Theorem 3.2] is done at the prime 2 for 1-connected spaces. Although this is probably well-known to the experts, we give here an alternative proof for this result that includes arbitrary connected spaces.

Lemma 2.1. *Let X be a p -complete, connected space of finite type. Then $QH^*(X; \mathbb{F}_p)$ is locally finite if and only if ΩX is $B\mathbb{Z}/p$ -local.*

Proof. By [21, Proposition 3.9.7 and 6.4.5] $QH^*(X; \mathbb{F}_p)$ is locally finite if and only if $T_V(H^*(X; \mathbb{F}_p))_{T_V(c)} \cong H^*(X; \mathbb{F}_p)$ for any elementary abelian p -group V . By [13, Proposition 3.4.4], this is so if and only if Lannes' T functor computes the cohomology of $\text{map}(BV, X)_c$. Therefore the above isomorphism can be restated by saying that $\text{map}(BV, X)_c \simeq X$, i.e. ΩX is $B\mathbb{Z}/p$ -local. ▮

Observe that many interesting spaces verify the condition that the module of indecomposables is locally finite. Let us mention one class of examples taken from Lannes, [13]. The second part of the proposition is a consequence of [13, Proposition 3.4.3] which states that the T -functor computes the cohomology of the corresponding mapping space. Recall that a group verifies *virtually* a certain property if it admits a subgroup of finite index which verifies the property.

Proposition 2.2. [13, p. 203] *Let G be a group of virtually finite mod p cohomological dimension. Then there is an isomorphism $TH^*(G; \mathbb{F}_p) \cong \prod_{\rho \in \text{Rep}(\mathbb{Z}/p, G)} H^*(C_G(\rho); \mathbb{F}_p)$. If moreover G has virtually finite mod p cohomology, then there is a weak equivalence $\text{map}(B\mathbb{Z}/p, BG)_p^\wedge \simeq \text{map}(B\mathbb{Z}/p, (BG)_p^\wedge)$.* ▮

Corollary 2.3. *Let G be a group of virtually finite mod p cohomology. The space $\Omega((BG)_p^\wedge)$ is then $B\mathbb{Z}/p$ -local.*

Proof. Consider the fibration $\text{map}_*(B\mathbb{Z}/p, (BG)_p^\wedge) \rightarrow \text{map}(B\mathbb{Z}/p, (BG)_p^\wedge) \rightarrow (BG)_p^\wedge$. By the above result of Lannes we know that the total space is equivalent to the p -completed mapping space $\text{map}(B\mathbb{Z}/p, (BG)_p^\wedge)$, which can be in turn identified with the p -completion of $\coprod BC_G(\rho)$, where the disjoint union is taken over the representations $\text{Rep}(\mathbb{Z}/p, G)$, see for example [7, Proposition 7.1]. The base point here is given by the trivial representation, i.e. lies in the classifying space of the trivial representation, whose centralizer is G itself. Therefore we obtain by looping once the above fibration that $\text{map}_*(B\mathbb{Z}/p, \Omega(BG)_p^\wedge)$ is contractible. \blacksquare

Example 2.4. A virtually nilpotent group G is by definition an extension of a finite group Q by a nilpotent group N . We notice first that if G is finitely generated one can always assume that N is torsion free since any finitely generated nilpotent group is virtually torsion free. Finitely generated torsion free nilpotent group have finite cohomological dimension, see for example [9, VIII.2]. We infer from the above corollary that $\Omega(BG)_p^\wedge$ is $B\mathbb{Z}/p$ -local for any finitely generated virtually nilpotent group G .

Consider the inclusion $O^p(Q) \rightarrow Q$ of the maximal p -perfect subgroup of Q as in [15] and construct the pull-back $G' = \text{lim}(O^p(Q) \rightarrow Q \leftarrow G)$. Since the quotient $Q/O^p(Q)$ is a p -group P , the fibration $BO^p(Q) \rightarrow BQ \rightarrow BP$ is preserved by p -completion, and so is the pull-backed one $BG' \rightarrow BG \rightarrow BP$. Therefore $(BG)_p^\wedge$ is the total space of a fibration

$$(BG')_p^\wedge \longrightarrow (BG)_p^\wedge \longrightarrow BP$$

where P is a finite p -group and G' has a normal, finitely generated, torsion-free, nilpotent subgroup N such that the quotient is p -perfect.

Remark that when G is virtually nilpotent, finitely generated, and torsion free, then BG is an *infra-nilmanifold* so that the cohomology of G itself is finite dimensional. In this case $(BG)_p^\wedge$ is $B\mathbb{Z}/p$ -local (and so is its loop space of course).

The class of groups of finite virtual cohomological dimension is much larger than that of virtually nilpotent ones, but we do not know if they are all \mathbb{F}_p -good, which

prevents us from being able to obtain our results in full generality. A well-known class of \mathbb{F}_p -good spaces is that with p -perfect fundamental group [5, Proposition 3.2].

Example 2.5. Let G be a p -perfect group of virtually finite mod p cohomology. Then $(BG)_p^\wedge$ is simply connected and $\Omega((BG)_p^\wedge)$ is $B\mathbb{Z}/p$ -local. Examples of such groups are given by the special linear groups $SL_n(\mathbb{Z})$ and the Steinberg groups $St_n(\mathbb{Z})$, which are even perfect groups. The homotopy groups of their p -completed classifying spaces are closely related to the algebraic K -theory groups of \mathbb{Z} .

Our next example is a slight generalization of the above. Recall that the *lower p -central series* of a group G is defined inductively by $\Gamma_0^p(G) = G$ and $\Gamma_{n+1}^p(G)$ is generated by elements of form $xyx^{-1}y^{-1}z^p$ for $x \in G$ and $y, z \in \Gamma_n^p(G)$. In particular a group G is p -perfect if and only if $G = \Gamma_1^p(G)$.

Example 2.6. In analogy with the terminology used by Wagoner in [23] and Loday in [17], we say that a group G is *quasi p -perfect* if the subgroup $\Gamma_1^p(G)$ is p -perfect. This means that G is an extension of an elementary abelian p -group with a p -perfect one. To make sure that BG is \mathbb{F}_p -good we impose the following condition:

For any finite set g_1, \dots, g_n of elements in $\Gamma_1^p(G)$ and $g \in G$ there exists an element $h \in \Gamma_1^p(G)$ such that $gg_i g^{-1} = hg_i h^{-1}$ for all $1 \leq i \leq n$.

This actually converts $(BG)_p^\wedge$ into a simple space, compare with [23, Lemma 1.3]. If one requires that G has virtually finite mod p cohomology, one obtains new examples of groups G such that $\Omega((BG)_p^\wedge)$ is $B\mathbb{Z}/p$ -local.

Example 2.7. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group, as defined by Broto, Levi, and Oliver in [8, Definition 1.8] and consider its classifying space $|\mathcal{L}|$. We know from [8, Theorem 5.8] that $H^*(|\mathcal{L}|; \mathbb{F}_p)$ is noetherian and can be computed by stable elements, just like the cohomology of an ordinary finite group. Therefore the module of indecomposables is finite and hence $\Omega(|\mathcal{L}|_p^\wedge)$ is $B\mathbb{Z}/p$ -local by Lemma 2.1.

3. THE DWYER-WILKERSON THEOREM

We recall in this section the theorem of Dwyer and Wilkerson about homotopy groups of 2-connected spaces with locally finite module of indecomposables. We

explain then how it can be efficiently applied to understand certain spaces which are not 2-connected by considering their 2-connected cover.

Theorem 3.1. [11, Theorem 1.3] *Let X be a 2-connected CW-complex of finite type with non-trivial reduced mod p cohomology such that the module $QH^*(X; \mathbb{F}_p)$ of indecomposable elements is locally finite. Then there exist infinitely many integers k such that $\pi_k X$ contains p -torsion.* ▮

The following elementary lemma is the key to understand which are the spaces which make it impossible to relax the connectivity assumption in the theorem.

Lemma 3.2. *Let $X = K(A, 2)$ be a p -complete space of finite type such that ΩX is $B\mathbb{Z}/p$ -local. Then A is isomorphic to a finite direct sum of copies of \mathbb{Z}_p^\wedge .*

Proof. Obviously A must be p -torsion free since $\Omega X \simeq K(A, 1)$ is assumed to be $B\mathbb{Z}/p$ -local. As X has finite type we infer that $H_1(A; \mathbb{F}_p)$, which is isomorphic to $H_2(X; \mathbb{F}_p)$, is finite. Hence by [4, Lemma 7.5] A is an abelian p -torsion free Ext- p -complete group of finite type, i.e. A is isomorphic to a finite direct sum of copies of \mathbb{Z}_p^\wedge (use Harrison's classification [5, VI.4.5] or Bousfield's comment on p -adically polycyclic groups in [4, p. 347]). ▮

Theorem 3.3. *Let X be a p -complete CW-complex of finite type such that the module $QH^*(X; \mathbb{F}_p)$ of indecomposable elements is locally finite. Then one of the following properties is satisfied:*

- (1) X is aspherical,
- (2) $X\langle 1 \rangle$ is a $K(\mathbb{Z}_p^\wedge, 2)^n$,
- (3) X has infinitely many homotopy groups with p -torsion.

In the last case moreover the space ΩX has infinitely many non-trivial k -invariants.

Proof. Let Y be the universal cover of X . Let us assume that X is not aspherical, and consider the 2-connected cover $Y\langle 2 \rangle$ of Y , which can be seen as total space in a fibration of spaces of finite type

$$K(\pi_2 Y, 1) \rightarrow Y\langle 2 \rangle \rightarrow Y$$

If we loop once this fibration, we know that ΩY is a $B\mathbb{Z}/p$ -local space by [11, Theorem 3.2]. Therefore $\Omega(Y\langle 2 \rangle)$ must be $B\mathbb{Z}/p$ -local as well since $\Omega K(\pi_2 Y, 1)$ is homotopically discrete. This means precisely that the module of indecomposables $QH^*(Y\langle 2 \rangle; \mathbb{F}_p)$ is locally finite. From Theorem 3.1 we infer that $Y\langle 2 \rangle$ (completed at p) is either contractible or has infinitely many homotopy groups with p -torsion, i.e. Y itself has infinitely many homotopy groups with p -torsion unless its p -completion is an Eilenberg-Mac Lane space of type $K(A, 2)$. In this case we infer from Lemma 3.2 that A is isomorphic to a finite direct sum of copies of \mathbb{Z}_p^\wedge .

The statement about the k -invariants is a direct consequence of Proposition 2.3. Indeed if the loop space ΩX only has a finite number of non-trivial k -invariants there exists an integer N such that the p -complete Eilenberg-Mac Lane space of finite type $K(\pi_n(\Omega X), n)$ is a retract for any $n \geq N$. Therefore this Eilenberg-Mac Lane space is $B\mathbb{Z}/p$ -local as well, which is only possible if $n \leq 2$. This implies that all higher homotopy groups are trivial and so X is aspherical by the first part of the theorem. \blacksquare

Since an unstable algebra which is locally finite as a module over the Steenrod algebra obviously has also a locally finite module of indecomposable, the Dwyer-Wilkerson condition truly generalizes the previously handleable cases. It is in fact straightforward to obtain the Lannes-Schwartz theorem as a corollary.

Corollary 3.4. *Let X be a simply connected CW-complex of finite type with non-trivial and locally finite mod p cohomology. Then there exists an infinite number of integers k such that $\pi_k X$ contains p -torsion.*

Proof. Since X is a $B\mathbb{Z}/p$ -local space, so is its loop space ΩX . The previous proposition applies and we conclude because the cohomology of $K(\mathbb{Z}, 2)$ is not locally finite. \blacksquare

In view of Theorem 3.3 a good understanding of the Dwyer-Wilkerson statement for arbitrary connected spaces goes through – compare with condition (2) – the study of 2-stage Postnikov pieces. In Lemma 3.2 we have identified the second homotopy group. As for the fundamental group we will assume that X is an \mathbb{F}_p -good space, so X_p^\wedge is p -complete.

By [10, Proposition 3.4] such spaces include all virtually nilpotent spaces (the action of the fundamental group on any homotopy group is virtually nilpotent). Bousfield characterizes the $H\mathbb{F}_p$ -local spaces in [3, Theorem 5.5] in terms of their homotopy groups, which implies in particular that the n -connected covers and the n -th Postnikov sections of p -complete (and \mathbb{F}_p -good) spaces are p -complete.

In short if X is a virtually nilpotent space, its p -completion X_p^\wedge is an $H\mathbb{F}_p$ -local space, and its second Postnikov section $Y = X_p^\wedge[2]$ is a p -complete space with only two homotopy groups. It can be seen as the total space of a fibration of the form

$$K(A, 2) \longrightarrow Y \longrightarrow K(G, 1)$$

where both $K(A, 2)$ and $K(G, 1)$ are p -complete spaces.

Lemma 3.5. *Let X be a virtually nilpotent space of finite type. Then $\pi_1(X_p^\wedge)$ is a p -complete group isomorphic to $(\pi_1 X)_p^\wedge$. It is an extension of a finite p -group by a nilpotent p -complete one.*

Proof. The fundamental group G of X_p^\wedge is isomorphic to that of $K(G, 1)_p^\wedge$ by the Whitehead type theorem [4, Proposition 4.1]. As the fundamental group of X is a finitely generated virtually nilpotent group, it is in particular polycyclic-by-finite. We conclude by Bousfield \mathbb{F}_p -goodness result [4, Theorem 7.2] on polycyclic-by-finite spaces that $\pi_1(X_p^\wedge) \cong G_p^\wedge$.

It remains to describe this virtually nilpotent group. As in Example 2.4 we can find normal subgroups $N \leq G' \leq G$ such that N is nilpotent, finitely generated, and torsion free, the quotient $Q = G'/N$ is p -perfect, and G/G' is a finite p -group. We will actually show that the inclusion $N \rightarrow G'$ induces an epimorphism $N_p^\wedge \rightarrow G_p'^\wedge$. By [4, Lemma 5.2] we only need to check that it induces an epimorphism on the first mod p homology group, i.e. the quotient by the first term of the mod p lower central series. Notice that the quotient $N/N \cap \Gamma_1^p G'$ is isomorphic to $G'/\Gamma_1^p G'$ because Q is p -perfect. Therefore the maximal quotient of N which is an elementary abelian group is at least as large as $H_1(G'; \mathbb{F}_p)$ and we are done. ▀

Summing up this result with Lemma 3.2 we can now describe quite accurately the p -complete 2-stage Postnikov pieces which have a $B\mathbb{Z}/p$ -local loop space.

Proposition 3.6. *Let X be a virtually nilpotent space of finite type with $\pi_n(X) = 0$ for any $n \geq 3$. Assume that $QH^*(X; \mathbb{F}_p)$ is locally finite. Then $\pi_1(X_p^\wedge)$ is isomorphic to $(\pi_1 X)_p^\wedge$, an extension of a finite p -group by a nilpotent p -complete group, and $\pi_2(X_p^\wedge)$ is isomorphic to a finite direct sum of copies of \mathbb{Z}_p^\wedge . \blacksquare*

When the fundamental group is finite, it must be a p -group and we recover precisely the class of 2-stage Postnikov systems studied by Grodal in [12].

4. UNIVERSAL COVERS OF p -COMPLETED SPACES

In this section we identify explicitly for four different families of spaces the universal covers of the p -completions, as p -completions of spaces inside the same family. We start by reminding the well-known case of finite groups, see [15], even though the next examples also contain all finite groups.

4.1. Finite groups. Let G be a finite group and $O^p(G)$ the maximal p -perfect subgroup of G . This is a normal subgroup and the quotient $P = G/O^p(G)$ is a p -group. Therefore the fibration $BO^p(G) \rightarrow BG \rightarrow BP$ is preserved by p -completion. Since $O^p(G)$ is p -perfect $(BO^p(G))_p^\wedge$ is simply connected and thus is weakly equivalent to the universal cover of $(BG)_p^\wedge$.

Hence for any classifying space of a finite group, the universal cover can be chosen, up to p -completion, to be another classifying space.

4.2. p -local finite groups. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group as in Example 2.7. We learn from [6, Theorem 4.4] that there exists a p -local finite group $(O^p(S), O^p(\mathcal{F}), O^p(\mathcal{L}))$ such that $|O^p(\mathcal{L})|_p^\wedge$ is the universal cover of $|\mathcal{L}|_p^\wedge$.

Again we see that the universal cover can be chosen, up to p -completion, inside the class of p -local finite groups.

4.3. Virtually nilpotent groups. For virtually nilpotent groups the idea to construct the universal cover of $(BG)_p^\wedge$ out of group theoretical information is already present in the work of Bastardas, [1, Section 5.3].

Let G be a finitely generated virtually nilpotent group. As in Example 2.4 we can find normal subgroups $N \leq G' \leq G$ such that N is nilpotent, finitely generated, and torsion free, the quotient $Q = G'/N$ is p -perfect, and G/G' is a finite p -group. Hence

$\pi_n(BG)_p^\wedge \cong \pi_n(BG')_p^\wedge$ for all $n \geq 2$. As we only wish to identify the universal cover, we might as well assume from now on that G sits in an extension

$$N \longrightarrow G \longrightarrow Q$$

where Q is finite and p -perfect, and N is a nilpotent group, finitely generated, and torsion-free. Recall from Lemma 3.5 that $\pi_1(BG)_p^\wedge \cong G_p^\wedge$ is a nilpotent group since Q is p -perfect.

Lemma 4.1. *Consider the fiberwise p -completion $N_p^\wedge \rightarrow \bar{G} \rightarrow Q$. The p -completion homomorphism $\bar{G} \rightarrow \bar{G}_p^\wedge$ is then surjective.*

Proof. The extension $N \rightarrow G \rightarrow Q$ gives rise to a fibration $BN \rightarrow BG \rightarrow BQ$, which can be fiberwise p -completed. Because $(BN)_p^\wedge$ is weakly equivalent to the classifying space of N_p^\wedge , the total space X of the new fibration is the classifying space of some group \bar{G} . We have proven in Lemma 3.5 that the composite $N_p^\wedge \rightarrow \bar{G} \rightarrow \bar{G}_p^\wedge$ is surjective, thus so is $\bar{G} \rightarrow \bar{G}_p^\wedge$. ■

Let us define now K as the kernel of the completion morphism $\bar{G} \rightarrow \bar{G}_p^\wedge$ (the intersection of all the terms in the mod p lower central series). In Theorem 4.3 we identify the universal cover of $(BG)_p^\wedge$ as the p -completed classifying space of the group K .

Proposition 4.2. *There is a fibration $(BK)_p^\wedge \longrightarrow (B\bar{G})_p^\wedge \longrightarrow B(G_p^\wedge)$.*

Proof. We apply the “nilpotent action lemma” [10, 5.1] of Dwyer, Farjoun, and Kan to the pair of fibrations $BK \rightarrow B\bar{G} \rightarrow B(G_p^\wedge)$ and $(BN)_p^\wedge \rightarrow B\bar{G} \rightarrow BQ$. We deduce from Lemma 4.1 that they induce an epimorphism $\bar{G} \rightarrow G_p^\wedge \times Q$. Because the p -completion of a nilpotent group is nilpotent again, we deduce that the nilpotent group G_p^\wedge acts nilpotently on all homology groups $H_n(BK; \mathbb{F}_p)$. Therefore the nilpotent fibration lemma [10, Proposition 4.2(i)] applies and we are done. ■

Theorem 4.3. *Let G be a virtually nilpotent group which is an extension of a finite, p -perfect group by a nilpotent, finitely generated, and torsion-free one. The space $(BK)_p^\wedge$ is the universal cover of $(BG)_p^\wedge$.*

Proof. One knows that the fundamental group of $(BG)_p^\wedge$ is G_p^\wedge ([4, Theorem 7.2]). In view of the above proposition we only need to remark that $BG \rightarrow B\bar{G}$ is an $H\mathbb{F}_p$ -equivalence (the p -completion $N \rightarrow N_p^\wedge$ is so for N is nilpotent). \blacksquare

We note that K is virtually nilpotent as well, being a subgroup of a virtually nilpotent one. We can even say more, since N_p^\wedge is the p -completion of a finitely generated torsion free nilpotent group: K is p -adically polycyclic-by-finite. In the situation where G is actually finite, K coincides with $O^p(G)$, which is consistent with the approach of Levi, compare with 4.1.

4.4. Quasi p -perfect groups. By definition, see Example 2.6, a group G is quasi p -perfect if the subgroup $\Gamma_1^p(G)$ is p -perfect. The fibration

$$B\Gamma_1^p(G) \longrightarrow BG \longrightarrow BH_1(G; \mathbb{F}_p)$$

is preserved under p -completion since $H_1(G; \mathbb{F}_p)$ is an (elementary abelian) p -group and BG is \mathbb{F}_p -good. Hence the universal cover of $(BG)_p^\wedge$ is $(B\Gamma_1^p(G))_p^\wedge$.

5. HOMOTOPY GROUPS OF p -COMPLETED CLASSIFYING SPACES

We obtain in this section in a single proof the results which were known before about p -completions of classifying spaces of finite groups (Levi) and virtually nilpotent groups (Bastardas). We prove along the same lines a new result for quasi p -perfect groups and p -local finite groups.

Theorem 5.1. *Let X be the classifying space of a member of the following four families:*

- (1) *finite groups,*
- (2) *p -local finite groups,*
- (3) *finitely generated virtually nilpotent groups,*
- (4) *quasi p -perfect groups of finite virtual mod p cohomology.*

Then the p -completion of X is either aspherical or it has infinitely many homotopy groups with p -torsion. In this case moreover the space $\Omega(X)_p^\wedge$ has infinitely many non-trivial k -invariants.

Proof. In view of the previous section it remains to prove that, up to p -completion, $K(\mathbb{Z}, 2)$ cannot belong to any of the four families. Recall that its mod p cohomology is polynomial on a generator in dimension 2, hence concentrated in even dimensions. Therefore there are no Bocksteins at all, which means by an elementary Bockstein spectral sequence argument that there is no p -torsion in the higher integral homology groups.

We wish now to use a transfer argument to show that this cannot be the cohomology of a space in any of the four families. When X is the classifying of a finite group, choose Y to be the classifying space of the trivial subgroup. When X is the classifying space of a virtually nilpotent group, we can assume as in 4.3 that $X = BK$ and so X_p^\wedge is simply connected. Choose then Y to be the classifying space of the nilpotent subgroup of finite index $K \cap N_p^\wedge$, a subgroup of a group of finite homological dimension, [9, VIII.2]). Finally when X is the classifying space of a quasi p -perfect group of virtually finite mod p cohomology, we choose Y to be the classifying space of some subgroup of finite index which has finite mod p cohomology. A standard transfer argument shows now that in all three cases the multiplication by the (finite) index is zero on high enough integral homology groups of X .

In case (2) we use the recent work [20] of Ragnarsson to get a transfer for p -local finite groups. The mod p cohomology of a p -local finite group is contained as a retract in the cohomology of its Sylow p -subgroup as a unstable subalgebra over the Steenrod algebra [20, Proposition 9.4]. The cohomology of $K(\mathbb{Z}, 2)$ which has no (higher) Bocksteins cannot be a retract of the cohomology of a finite group. \blacksquare

We wish to point out that the assumptions made on the virtually nilpotent group could hardly be relaxed. For example if one drops the finitely generated hypothesis, the result is obviously false, as shown by the well-known example of the Prüfer group: $(B\mathbb{Z}/p^\infty)_p^\wedge \simeq K(\mathbb{Z}_p^\wedge, 2)$.

Let us mention that the p -completed classifying spaces of p -perfect groups of finite virtual cohomological dimension have been studied by Levi. He proves in [16, Theorem 1.4] that $\Omega(BG)_p^\wedge$ is a retract of some finite complex (which does not imply our result).

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