I. Galois meets Hopf

Recall: A field extension \( k \hookrightarrow E \) is Galois if it's algebraic, normal and separable. If \([E : k] < \infty\), then the extension is Galois if it's algebraic and \( k = \mathbb{E}(\text{Aut}_{E}(E)) \), the Galois group of the extension.

First generalizations:

- [Auslander-Goldman, 1960]: generalization to commutative rings
- [Chase-Harrison-Rosenberg, 1965]: six characterizations of Galois extensions of commutative rings, including:

\[ R \hookrightarrow S \text{ is } G\text{-Galois for } G \leq \text{Aut}_R(S), |G| < \infty \]

\[ \iff R \xrightarrow{G} S^G \text{ and } S \otimes_R S \xrightarrow{G} T_S \]

\[ s \otimes s' \mapsto (s \cdot g(s'))_{g \in G} \]

Beyond group actions (and on to group schemes...)

- [Chase-Sweedler, 1969], [Kreimer-Takeuchi, 1981]:

\( k \) = commutative ring, \( \otimes = \otimes_k \)

\( H = k\text{-bialgebra} \)

\( B = k\text{-algebra with coaction } \rho : B \rightarrow B \otimes H \)

\( A = B^{coH} = \{ b | \rho(b) = b \otimes 1 \} \) algebra homomorphism, coassociative, counital
The extension of \( k \)-algebras \( A \hookrightarrow B \) is \( H \)-Hopf-Galois if
\[
B \otimes_A B \xrightarrow{B \otimes p} B \otimes B \otimes H \xrightarrow{\mu \otimes H} B \otimes H
\]
is an isomorphism. (the Galois map)

**Examples:**

1. \( k \hookrightarrow E \) field extension, \( G \leq \text{Aut}_k(E) \), \( |G| < \infty \), \( F = E^G \):
\[
F \hookrightarrow E \text{ is } G \text{-Galois}
\]
\[
k^G = \text{Hom}_k(k[G], k)
\]
\[
F \hookrightarrow E \text{ is } k^G \text{-Hopf-Galois}
\]

2. \( X \) finite set, \( G \) finite group, \( a : X \times G \rightarrow X \) action, \( q : X \rightarrow X_G = Y \), \( k \) field:
\[
X \times G \xrightarrow{\Delta \times G} X \times X \times G \xrightarrow{X \times a} X \times X \quad (*)
\]
\[
\Rightarrow \text{ extension } k^Y \xrightarrow{q^*} k^X \otimes k^G \text{ - coaction}
\]
\[
q \text{ is a } k^G \text{-Hopf-Galois extension}
\]
\[
(*) \text{ is an isomorphism}
\]
\[
a \text{ is a free } G \text{-action}
\]

3. \( H \) a \( k \)-bialgebra:
\[
k \hookrightarrow H \text{ is } H \text{-Hopf-Galois}
\]
\[
H \text{ is a Hopf algebra}
\]
More generally: a Hopf algebra \( A \otimes H \) is a \( H \)-Hopf-Galois algebra of normal basis type.

Why interesting?
- Faithfully flat \( H \)-extensions over the coordinate ring of an affine group scheme correspond to \( G \)-principal bundles (torsors).
- Can study Hopf algebras via associated \( H \)-extensions.

II. Grothendieck Framework

\[ \varphi : A \rightarrow B \text{ ring homomorphism} \]
\[ \Rightarrow \text{adjunction } \otimes_A B : \text{Mod}_A \xleftrightarrow{\sim} \text{Mod}_B \varphi^* \]

Informal Grothendieck descent problem

@ Given \( N_B \), when \( \exists M_A \) such that \( N \cong M_A \)?
@ Given \( f : M_A \otimes B \rightarrow M'_A \otimes B \), when \( \exists g : M \rightarrow M' \) homomorphism of \( A \)-modules s.t. \( f = g \otimes_A M' \)?

More formally

\[ D(\varphi) = \text{category of descent data associated to } \varphi \]

Objects = pairs \( (N, \Theta) \) with \( N \in \text{Mod}_B \),
\[ \Theta : N \rightarrow \varphi^*(N) \otimes_A B \text{ - coassociative, counital} \]
3. Factorization: \[ \text{Mod}_A \xrightarrow{\otimes_A B} \text{Mod}_B \]
\[ \xrightarrow{\text{Can}} \xrightarrow{\mathcal{D}(\phi)} \text{Forget} \]
\[ (M \otimes_A B, \theta_M) \]

\[ M \otimes_A B \cong M \otimes_A A \otimes_A B \]
\[ \xrightarrow{M \otimes_A \phi \otimes_A B} (M \otimes_A B) \otimes_A B \]
\[ \theta_M \]

\( \phi \) satisfies effective Grothendieck descent if \( \text{Can}: \text{Mod}_A \rightarrow \mathcal{D}(\phi) \) is an equivalence.

\( \phi \) satisfies effective Grothendieck descent

\[ \implies \text{have answers to } @ \text{ and } \otimes \text{ can realize objects and morphisms in } \text{Mod}_B \text{ when they underlie objects and morphisms in } \mathcal{D}(\phi). \]

\[ III. \text{Quillen} \]

"Up-to-homotopy" versions of Hopf-Galois and Grothendieck theories.

Motivation:

[Roigne, 2008]: Galois theory of structured ring spectra.

One important extension that is not Galois but is Hopf-Galois:

\[ S \rightarrow M \mathbb{U} \]

"Galois" and "Hopf-Galois" interpreted homotopically. Isomorphisms...
Grothendieck descent "up-to-homotopy" for morphisms of structured ring spectra also important, e.g., for studying completions.

**Framework**

- $(N, \wedge, S)$ monoidal model category (nice enough)
- $\phi: A \rightarrow B$ morphism of monoids in $N$
- $H$ bimonoid in $N$
- $\rho: B \rightarrow B \wedge H$ coaction st.

\[ A \overset{\phi}{\rightarrow} B \wedge H \]

**Notation:** $A \overset{\phi}{\rightarrow} B^{\wedge H}$

**Hopf-Galois data**

**Schwede-Shipley:** Well understood conditions under which $\exists$ Quillen model category structure on $\underline{\text{Mod}}_A, \underline{\text{Mod}}_B, \underline{\text{Alg}}$.

**[H.-Shipley], [BHKKRS]:**

New "left-induction" techniques $\Rightarrow$ reasonable conditions guaranteeing existence of Quillen model category structure on $\underline{\Omega}(\phi), \underline{\text{Alg}}^H$.

**Defn:** $A \overset{\phi}{\rightarrow} B^{\wedge H}$ is a homotopic Hopf-Galois extension if:

- $A \overset{\phi}{\rightarrow} B^{\wedge H}$ Need model category structure on $\underline{\text{Alg}}^H$ to define this.

\[ B \underset{A}{\wedge} B \overset{\phi \otimes_H}{\rightarrow} B \wedge H \overset{\mu^H}{\rightarrow} B^{\wedge H} \]

is a weak equivalence.
**Defn.** \( \varphi : A \to B \) satisfies effective homotopic Grothendieck descent if

\[ \text{Can: } \text{Mod}_A \to \mathcal{D}(\varphi) \]

is a Quillen equivalence.

**Example:** \( \mathbb{k} \)-commutative ring,

\( H \) - 1-connected dg \( \mathbb{k} \)-bialgebra,

degree-wise \( \mathbb{k} \)-projective

\( E \) - dg \( H \)-comodule algebra

\( [H \text{-Levi}]: \Omega_2(-; H; -) : \text{Alg}^H \times H \text{Alg} \to \text{Alg} \)

- the two-sided cobar construction

**Proposition:** [Berglund - H.]

\[ \Omega(E; H; \mathbb{k}) \to \Omega(E; H, H) \]

Homotopic normal basis extension

is homotopic \( H \)-Hopf-Galois and satisfies effective homotopic Grothendieck descent.

It's not a fluke that this morphism both is HG and satisfies descent!

**IV. All together now! \[ \varphi = 
\]

**Theorem:** Let \( H \) be as above. Let \( \varphi : A \to B^H \) be \[ \text{Berglund-H.} \]

Hopf-Galois data such that \( A \to B^H \)

Then:

\( \varphi \) is htpic \( H \)-Hopf-Galois \( \iff \) \( \varphi \) satisfies effective htpic Grothendieck descent.

| Proof by reducing to normal extension. |
Remarks: • Schneider proved a result with a similar flavor in the classical context.
  • Rognes proved analogous results for commutative ring spectra.

And Koszul?

Recall: A Koszul algebra $A$ has a Koszul dual coalgebra $C$ such that $A \cong \text{Mod}_A \cong \text{Comod}_C$.

Corollary: Given a $A \rightarrow B$ as above.

If $B \cong \mathbb{k}$, then $H$ is a generalized Koszul dual of $B$, i.e.,

$\text{Ho}(\text{Mod}_B) \cong \text{Ho}(\text{Comod}_H)$.

Example: $E \cong \mathbb{k}$ ⇒ $H$ is a generalized Koszul dual of $\Omega(E, \mathbb{H}, \mathbb{k})$. 
