I. Descent from a homotopical viewpoint

A. Classical descent theory [Mesablishvili]

\( \phi: A \to B \) ring homomorphism

\[ \Rightarrow \text{adjunction} \quad - \otimes_A B : \text{Mod}_A \rightleftarrows \text{Mod}_B : \phi^* \]

Informal descent problem (Realizability!)

(a) Given \( N \in \text{Mod}_B \), under what conditions \( \exists M \in \text{Mod}_A \) such that \( N \cong M \otimes_A B ? \)

(b) Given \( f: M \otimes_A B \to M' \otimes_A B \), under what conditions \( \exists g: M \to M' \)

The formal framework

* The descent co-ring associated to \( \phi \):

\[ W_\phi = (B \otimes_A B, \delta_\phi, \epsilon_\phi) \quad \text{(explain!)} \]

* \( \mathcal{D}(\phi) = \) the category of descent data for \( \phi \)

  * \( \text{Ob} \ \mathcal{D}(\phi) = W_\phi \)-comodules in \( \text{Mod}_B \)

  * \( \text{Mor} \ \mathcal{D}(\phi) : \) preserve obvious structure

* Functors: Extension of scalars factors through \( \mathcal{D}(\phi) \)

\[ \text{Mod}_A \xrightarrow{- \otimes_A B} \text{Mod}_B \]

\[ M \xleftarrow{\text{Can}_\phi} \mathcal{D}(\phi) \xrightarrow{u} (M \otimes_A B, \rho_M) \]

Rmv: Can be seen as a Tannakian realization problem wrt fiber functor \(- \otimes_A B\)
*Defn:* $\mathcal{G}$ satisfies (effective) descent if $\text{Cone}_g$ is fully faithful (resp. an equivalence of categories).  

**Formal descent problem**

When does $\mathcal{G}$ satisfy descent (⇒ answer to (b))? Effective descent (⇒ answer to (a) as well)?

*Rmk:* $\mathcal{G}$ satisfies effective descent ⇒ $U : \mathcal{T}(\mathcal{G}) \to \text{Mod}_B$ solves the associated Tannakian realization problem, with associated "Galois co-group" $W_\mathcal{G}$.

2. **Classical monadic descent:** generalizing Grothendieck descent

**Categorical preliminaries**

A monad on a category $\mathcal{C}$ consists of

- an endofunctor $T : \mathcal{C} \to \mathcal{C}$,
- a natural transformation $\mu : T \circ T \to T$, and
- a natural transformation $\eta : \text{Id}_\mathcal{C} \to T$ such that

$$
\begin{align*}
T^3X & \xrightarrow{\mu_{TX}} T^2X \\
T(\mu_X) & \downarrow \quad \downarrow \mu_X
\end{align*}
$$

and

$$
\begin{align*}
T^2X & \xrightarrow{T\mu_X} TX \\
T\eta_X & \downarrow \quad \downarrow \eta_X
\end{align*}
$$

commute if $X \in \text{Ob}\mathcal{C}$.

**Notation:** $\mathcal{T} = (T, \mu, \eta)$

Dually, a comonad on $\mathcal{C}$ consists of $K : \mathcal{C} \to \mathcal{C}$,

$\Delta : K \to K^2$ and $\varepsilon : K \to \text{Id}_\mathcal{C}$ such that $\Delta K \circ \Delta = K \Delta \circ \Delta$ and

$\varepsilon \Delta = \text{Id}_K$.

**Notation:** $\mathcal{K} = (K, \Delta, \varepsilon)$
Exercise: Let \( L : \mathcal{B} \xrightarrow{\sim} \mathcal{C} : R \) be an adjunction. Show that \((RL, R\varepsilon_L, \eta)\) is a monad on \(\mathcal{B}\) and that \((LR, L\eta_R, \varepsilon)\) is a comonad on \(\mathcal{C}\), where 
\[ \eta : \text{Id}_\mathcal{B} \rightarrow RL \text{ and } \varepsilon : LR \rightarrow \text{Id}_\mathcal{C} \] are the unit and counit of the adjunction.

From (co)monads to adjunctions

- A monad on \(\mathcal{C}\) \(\Rightarrow\) category \(\mathcal{C}^\Pi\) of \(\Pi\)-algebras
  - With objects: \((A, m)\) where \(m : \Pi A \rightarrow A\) in \(\mathcal{C}\) st
    \[
    \begin{array}{ccc}
    \Pi^2 A & \xrightarrow{\Pi m} & \Pi A \\
    \downarrow \mu_A & & \downarrow m \\
    \Pi A & \xrightarrow{m} & A
    \end{array}
    \]
    and \(A \xrightarrow{\delta_A} \Pi A\) commute.

Exercise: Show that \((\Pi X, \mu_X)\) is a \(\Pi\)-algebra if \(X \in \text{Ob}\mathcal{C}\) and \((\Pi X, \mu_X)\) is the free \(\Pi\)-algebra on \(X\).

Show moreover that there is an adjunction

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Pi^*} & \mathcal{C}^\Pi \\
A \longleftarrow & & \longleftarrow (A, m)
\end{array}
\]

Exercise: Let \(\varphi : A \rightarrow B\) be a ring hom. Let \(\Pi \varphi\) denote the monad on \(\text{Mod}_A\) associated to the adjunction \((- \otimes_A B) \xrightarrow{\varphi^*} -\). Show that \((\text{Mod}_A)^\Pi \varphi \leq \text{Mod}_B\).

Exercise: Show that the monad associated to \((\star)\) is \(\Pi\).

Exercise: Let \(K^\Pi\) denote the comonad on \(\mathcal{C}^\Pi\) associated to \((\star)\). Show that \(K^\Pi (A, m) = (\Pi A, \mu_A)\), 
\[
(\Delta^\Pi)_{(A, m)} = \Pi \eta_A : (\Pi A, \mu_A) \rightarrow (\Pi^2 A, \mu_{\Pi A}) \text{ and } (\varepsilon^\Pi)_{(A, m)} = m : (\Pi A, \mu_A) \rightarrow (A, m).
\]
Exercise: Calculate $K^{q}$ explicitly.

Dually ...

- $K$ comonad on $C$ $\Rightarrow$ adjunction $C_{K} \xrightarrow{\text{F}_K} C$, where
  - $C_{K}$ is the category of $K$-coalgebras, with objects $(C,S)$, $S: C \to KC$ s.t. $\Delta_{C} \circ S = T S \circ S$ and $\varepsilon_{C} \circ S = \text{Id}_{C}$,
  - $F_{K} X = (KX, \Delta_{X})$ the cofree $K$-coalgebra on $X$
  - $U_{K} (C,S) = C$

Exercise: Let $K_{q}$ denote the comonad associated to $(- \otimes B)^{q}$ for some ring homomorphism $q: A \to B$. Show that

$$(\text{Mod}_{B})_{K_{q}} \equiv \mathcal{B}(q).$$

Informal monadic descent problem

(a) When is a $\mathcal{M}$-algebra isomorphic to a free $\mathcal{M}$-algebra?

(Realizability)

(b) Given $f: (TX, \mu_{X}) \to (TY, \mu_{Y})$ in $C^{\mathcal{M}}$, when is there $g: X \to Y$ in $C$ such that $f = Gg$?

The formal framework

- $X \xleftarrow{\text{can}^{T}} C_{\mathcal{M}} \xrightarrow{\text{F}_\mathcal{M}} C^{\mathcal{M}}$

- $\text{Can}^{\mathcal{M}}(TX, \mu_{X}, \eta_{X})(C_{\mathcal{M}})^{K_{\mathcal{M}}} = \mathcal{B}(\mathcal{M})$ - the category of $\mathcal{M}$-descent data.

(*) Exercise: Show that $\text{Can}^{\mathcal{M}}$ admits a right adjoint if $C$ admits equalizers: $\text{Prin}^{\mathcal{M}}(A,m,S) = \text{lim} (A \xleftarrow{S \cdot m} TA)$.  

Def: $\mathcal{M}$ satisfies descent (resp. effective descent) if $\text{Can}^{\mathcal{M}}$ is fully faithful (resp. an equivalence of categories).
Formal descent problem

When does $\mathcal{P}$ satisfy descent (⇒ answer to (b))? Effective descent (⇒ answer to (a) as well)?

Rmk: Descent data more highly structured than objects in $\mathcal{C}$ and therefore, in principle, easier to compute and classify. The descent framework is also well suited to the study of rigidity problems.

Sometimes we need to work object by object...

**Defn:** $\mathcal{P}$ satisfies descent at $Y$ if

$$\text{Can}_T : \mathcal{C}(X,Y) \xrightarrow{\cong} \mathcal{D}(\mathcal{P})(\text{Can}^\pi X, \text{Can}^\pi Y) \forall X \in \text{Ob} \mathcal{C}.$$

**Exercise:** An object $Y$ in $\mathcal{C}$ is $\mathcal{P}$-injective if $y_Y : Y \rightarrow \mathcal{T}Y$ admits a retraction $\rho : \mathcal{T}Y \rightarrow Y$. In particular,

$$(A,m) \in \mathcal{C}^\mathcal{P} \mapsto A \text{ $\mathcal{P}$-injective, e.g., } A=\mathcal{T}x!$$

Show that: $Y$ $\mathcal{P}$-injective $\Rightarrow$ $\mathcal{P}$ satisfies descent at $Y$,

as long as $\mathcal{C}$ admits equalizers.

**Hint:** Use the bijection

$$\mathcal{D}(\mathcal{P})(\text{Can}^\pi X, \text{Can}^\pi Y) \cong \mathcal{C}(X, \text{Prm}^\pi \text{Can}^\pi Y).$$

**Exercise:** Characterize those $A$-modules at which $\mathcal{P}_A$ satisfies descent, when $\mathcal{P} : A \rightarrow B$ is:

1. $\mathbb{Z} \rightarrow \mathcal{P}_\mathbb{Z}$ (reduction mod $p$)
2. $R \hookrightarrow R[x]$, for any commutative ring $R$
Dually...

Comonadic codescent

Let $K$ be a comonad on $C$. Consider the diagram:

\[
\begin{array}{c}
E_K \xrightarrow{\eta_K} E \\
\downarrow \quad \downarrow \\
C \xrightarrow{\text{Can}_{ik}^{\text{co}}} C
\end{array}
\]

Where

- $\text{Can}_{ik}^{\text{co}}(X) = (KX, \Delta_X, KE_X)$

- $(C, S, m) \in \mathcal{D}^{\text{co}}(K) \iff$

- $m \text{ is a retraction of } S \text{ in } E_K$.

Def: $K$ satisfies codescent (resp. effective codescent) if $\text{Can}_{ik}^{\text{co}}$ is fully faithful (resp. an equivalence of categories).

Exercise: Let $\phi : E \to B$ be a continuous map (e.g., \(\Pi \quad \eta \mapsto \eta \quad \phi \)). Consider the associated adjunction $\phi^*: \text{Top}/E \leftrightarrow \text{Top}/B$. Show that

- $\text{Top}/E \equiv (\text{Top}/B)_{K^\phi}$

- $\mathcal{D}(K^\phi) = (\text{Top}/E)^{\text{co}}_{K^\phi}$: objects can be viewed as $f : X \to E$ + $c : X \times E \to X \times B$ satisfying cooyde condition + normalization: gluing data!
B. Homotopic descent

Goal: Develop descent and co-descent theory in an "up-to-homotopy" form. A more elaborate version than that presented here, involving \( \mathcal{V} \)-model categories, has also been developed.

1) **Model categories of algebras and coalgebras**

Theorem: [Schwede-Shipley, 2000] Let \( \mathcal{M} \) be a cofibrantly generated model category with sets \( \mathcal{E} \) and \( \mathcal{M} \) of generating cofibrations and generating acyclic cofibrations, respectively. Let \( \mathcal{T} \) be a monad on \( \mathcal{M} \) such that \( \mathcal{T} \) commutes with filtered colimits.

If: i) the domains of \( F^\mathcal{T}(\mathcal{E}) \) and \( F^\mathcal{T}(\mathcal{M}) \) are small with respect to \( F^\mathcal{T}(\mathcal{E}) \)-cell and \( F^\mathcal{T}(\mathcal{M}) \)-cell, respectively, and

ii) \( U^\mathcal{T}(F^\mathcal{T}(\mathcal{M}) \text{- cell}) \subseteq \mathcal{W}_E \),

then \( \mathcal{M}^\mathcal{T} \) admits a cofibrantly generated model category structure with sets \( F^\mathcal{T}(\mathcal{E}) \) and \( F^\mathcal{T}(\mathcal{M}) \) of generating cofibrations. In particular,

\[
\mathcal{W}_E \mathcal{M}^\mathcal{T} = (U^\mathcal{T})^{-1}(\mathcal{W}_E \mathcal{M}) \quad \text{and} \quad \mathcal{Fib}_{\mathcal{M}^\mathcal{T}} = (U^\mathcal{T})^{-1}(\mathcal{Fib}_\mathcal{M}),
\]

whence

\[
F^\mathcal{T}: \mathcal{M} \to \mathcal{M}^\mathcal{T}: U^\mathcal{T}
\]

is a Quillen pair.

Remark: Hypothesis (ii) can be replaced by:

(iii) every object of \( \mathcal{E} \) is fibrant and every \( \mathcal{T} \)-algebra has a path object.

Before stating the existence theorem for model category structure on categories of coalgebras, need another category-theoretic notion.
**Def:** A category $\mathcal{C}$ is **locally presentable** if
\begin{itemize}
  \item[i)] $\mathcal{C}$ is small cocomplete;
  \item[ii)] $\exists$ set $S \subseteq \text{Ob} \mathcal{C}$ such that every object is the colimit of a diagram of objects in $S$;
  \item[iii)] every object in $S$ is small wrt $\text{Mor} \mathcal{C}$;
  \item[iv)] $\mathcal{C}(X,Y)$ is a set $\forall X,Y \in \mathcal{C}$.
\end{itemize}

**Def:** A model category is **combinatorial** if it is cofibrantly [Smith] generated and locally presentable.

**Examples:**
- $s\text{Set}$ with either Kan or Joyal model structure
- $d\text{Set}$
- $\text{Ch}^{\infty}_{\mathbb{R}}$ with the projective model structure

**Non-example:** $\text{Top}$ is cofibrantly generated but not combinatorial.

**Proposition:** [Adámek-Rosický] If $\mathcal{C}$ is locally presentable and $\mathcal{T}$ is a monad on $\mathcal{C}$ such that $\mathcal{T}$ preserves filtered colimits, then $\mathcal{C}^\mathcal{T}$ is also locally presentable.

**Corollary:** Under the hypotheses of the Schwede-Shipley theorem, if $\mathcal{M}$ is combinatorial, so is $\mathcal{M}^\mathcal{T}$.

What about categories of coalgebras?

**Observations:**
1) For any comonad $\mathcal{K}$ on a category $\mathcal{C}$, colimits in $\mathcal{C}^\mathcal{K}$ are created in $\mathcal{C}$. In particular, if $\mathcal{C}$ is cocomplete, so is $\mathcal{C}^\mathcal{K}$. (*Exercise!*)

2) Since $\mathcal{F}^\mathcal{K}$ is a right adjoint, it’s easy to calculate limits in $\mathcal{C}^\mathcal{K}$ of diagrams in the image of $\mathcal{F}^\mathcal{K}$, e.g., $\mathcal{F}^\mathcal{K}(X \times Y) \cong \mathcal{F}^\mathcal{K}(X) \times \mathcal{F}^\mathcal{K}(Y)$. But other limits??
**Lemma:** [Adámek-Rosicky] Let $K$ be a comonad on a well-powered (e.g., locally presentable) category $C$. If $K$ preserves monomorphisms, then $C_{K}$ is complete.

**Lemma:** [Barr-Wells] Let $K$ be a comonad on a complete category $C$. If $K$ commutes with countable inverse limits, then $C_{K}$ is complete.

The following construction dual to ( )-cell is crucial to proving the existence of model category structure for categories of coalgebras.

**Defn.** Let $\mathcal{X}$ be a class of morphisms in a complete category $C$. An $\mathcal{X}$-Pastnikov tower is the composition

$$\lim_{\beta < \lambda} Y_{\beta} \to Y_{0}$$

of a tower $Y_{\beta}: \lambda^{\mathcal{X}} \to C$ for some ordinal $\lambda$, i.e.,

$$\ldots \to Y_{\beta+1} \xrightarrow{q_{\beta+1}} Y_{\beta} \to \ldots \to Y_{2} \xrightarrow{q_{2}} Y_{1} \xrightarrow{q_{1}} Y_{0},$$

where for every $\beta < \lambda$, $\exists \phi_{\beta+1}: X_{\beta+1} \to X_{\beta} \in \mathcal{X}$ and $k_{\beta}: Y_{\beta} \to X_{\beta}$ such that

$$Y_{\beta+1} \xrightarrow{q_{\beta+1}} \downarrow \phi_{\beta+1} \downarrow \leftarrow \downarrow \downarrow k_{\beta} \downarrow \leftarrow \downarrow q_{\beta+1} \downarrow \leftarrow \downarrow Y_{\beta} \xrightarrow{k_{\beta}} X_{\beta}.$$

If $\beta$ is a limit ordinal, then $Y_{\beta} = \lim_{\gamma < \beta} Y_{\gamma}$.

**Notation:** $Post_{\mathcal{X}} = \{ \text{\emph{\$\mathcal{X}$}}\text{-}\text{Pastnikov towers}\}$.

- $\forall y \in \text{Mor} C \Rightarrow \hat{y} = \text{retract-closure of } y$. 
**Definition:** A Postnikov presentation of a class \( \mathcal{Q} \subseteq \text{Mor} \mathcal{M} \) in a category \( \mathcal{M} \) is a class \( \mathcal{Q} \) such that \( \mathcal{Q} = \text{Post} \mathcal{Q} \).

**Theorem:** [Bayeh-H-Karpova-Kedzior-Riehl-Shipley] Let \( \mathcal{M} \) be a combinatorial model category such that \( \text{Fib}_\mathcal{M} \cap \text{WE}_\mathcal{M} \) admits a Postnikov presentation via \( \mathcal{Q} \subseteq \text{Mor} \mathcal{M} \). Let \( K \) be a comonad on \( \mathcal{M} \) such that \( K \) preserves filtered colimits and monomorphisms or countable inverse limits. If \( U_{\mathcal{K}} (\text{Post} \mathcal{Q}_K) \subseteq \text{WE}_\mathcal{M} \), then \( \mathcal{M}_K \) admits a model category structure with

\[
\text{WE}_{\mathcal{M}_K} = U^{-1}_{\mathcal{K}}(\text{WE}_\mathcal{M}), \quad \text{Cof}_{\mathcal{M}_K} = U^{-1}_{\mathcal{K}}(\text{Cof}_\mathcal{M}) \quad \text{and} \quad \text{Fib}_{\mathcal{M}_K} \cap \text{WE}_{\mathcal{M}_K} = \text{Post} \mathcal{Q}_K.
\]

In particular, \( U_{\mathcal{K}} : \mathcal{M}_K \xrightarrow{\sim} \mathcal{M} : F_{\mathcal{K}} \) is a Quillen pair.

**Remark on the proof:** Relies heavily on a very recent result of Mukai and Rosicky, on the existence of left-induced weak factorization systems.

**Corollary:** Let \( \mathcal{M} \) be a combinatorial model category with sets \( \mathcal{E} \) and \( \mathcal{J} \) of generating cofibrations. Let \( \mathcal{P} \) be a monad on \( \mathcal{M} \) such that \( \mathcal{P} \) preserves filtered colimits and monomorphisms or countable inverse limits. Let \( \mathcal{Q} \subseteq \text{Mor} \mathcal{M}^{\mathcal{P}} \) satisfy \( (U^{\mathcal{P}})^{-1}(\text{Fib}_\mathcal{M} \cap \text{WE}_\mathcal{M}) = \text{Post} \mathcal{Q} \).

If

i) \( F^\mathcal{P}(\mathcal{E}), F^\mathcal{P}(\mathcal{J}) \) small wrt \( F^\mathcal{P}(\mathcal{E}) \)-cell, \( F^\mathcal{P}(\mathcal{J}) \)-cell, resp;

ii) \( U^\mathcal{P}(\mathcal{F}(\mathcal{J}) \text{-cell}) \subseteq \text{WE}_\mathcal{M} \), and

iii) \( U^\mathcal{P} U_{\mathcal{K}^\mathcal{P}} (\text{Post} F_{\mathcal{K}^\mathcal{P}}(\mathcal{Q})) \subseteq \text{WE}_\mathcal{M} \),

then \( \exists \) Quillen pair \( U_{\mathcal{K}^\mathcal{P}} : \mathcal{M}_K \xrightarrow{\sim} \mathcal{M}^{\mathcal{P}} : F_{\mathcal{K}^\mathcal{P}} \).
Remarks: 1) Both $\mathcal{D}(\mathbb{F})$ and $\mathcal{M}(\mathbb{F})$ are combinatorial, with

$$WE_{\mathcal{D}(\mathbb{F})} = (\mathcal{U}_{\mathcal{D}(\mathbb{F})})^{-1}(\mathcal{U}_{\mathcal{D}(\mathbb{F})})^{-1}(WE_{\mathbb{F}})$$

and

$$WE_{\mathcal{M}(\mathbb{F})} = (\mathcal{U}_{\mathcal{M}(\mathbb{F})})^{-1}(WE_{\mathbb{F}}).$$

2) [H-Shipley] Different conditions for existence of model category structure on $\mathcal{M}_{\mathbb{F}}$, relying on a sort of “stability” in $\mathbb{F}$ and a filtration of $WE_{\mathbb{F}}$ that is appropriately compatible with $\mathbb{F}$.

Example: Let $g : A \rightarrow B$ be a morphism of non-negatively graded, dg $\mathbb{F}$-algebras such that $- \otimes_A B$ preserves monomorphisms (i.e., degree-wise injective maps), e.g., $B$ is $A$-semifree as a left $A$-module. Since colimits in $\mathcal{M}_A$ and $\mathcal{M}_B$ are created in $\mathcal{Ch}_{\mathbb{F}}^{>0}$, $T_B = g^*(- \otimes_A B)$ preserves all colimits. Moreover, conditions i) and ii) can easily be seen to hold (cf. [Schwede-Shipley]). It remains thus only to determine conditions under which condition iii) also holds.

For example, if $B$ is $A$-semifree on a graded $\mathbb{F}$-module of finite type, then all limits in $\mathcal{D}(\mathbb{F}) = \mathcal{D}(g)$ are created in $\mathcal{M}_B$, so an argument based on the Mittag-Leffler condition shows easily that $\exists$ Quillen pair

$$U : \mathcal{D}(g) \leftarrow \rightarrow \mathcal{M}_B.$$

2) Homotopic descent

Henceforth: $\mathcal{M}$ a model category, $\mathbb{F}$ a monad on $\mathcal{M}$ such that $\exists$ diagram of Quillen pairs

\[
\begin{array}{ccc}
\mathcal{D}(\mathbb{F}) & \leftarrow \rightarrow & \mathcal{M}^\mathbb{F} \\
\mathcal{M} & \leftarrow \rightarrow & \mathcal{M} \end{array}
\]

with weak equivalences created in $\mathcal{M}$.
Recall/Defn: Let $\mathcal{M}$ be a model category. The simplicial localization of $\mathcal{M}$ is a simplicial enrichment of $\mathcal{M}$, denoted $\Map_{\mathcal{M}}((-,-))$, such that
\[
\begin{align*}
X' & \overset{\sim}{\longrightarrow} X, \\
Y & \overset{\sim}{\longrightarrow} Y',
\end{align*}
\implies \Map_{\mathcal{M}}(X,Y) \overset{\sim}{\longrightarrow} \Map_{\mathcal{M}}(X',Y').
\]
and $\pi_0 \Map_{\mathcal{M}}(X,Y) \cong \Ho(\mathcal{M})(X,Y) \cong [X^c, Y^c]$.
\Map_{\mathcal{M}}(X,Y)$ is the derived mapping space.

Remark: The simplicial localization of a model category always exists, and there are many weakly equivalent ways of constructing the derived mapping space, e.g., by [Dugger]:

\[\Map^h_{\mathcal{M}}(X,Y) = \text{Nerve}(X \xleftarrow{\sim} \xrightarrow{\sim} Y), \text{ if } X \text{ cofibrant.} \]

(Need to be a little careful about smallness…)

Defn: The monad $\Theta$ satisfies homotopic descent if
\[
\Map^h_{\mathcal{M}}(X,Y) \overset{\sim}{\longrightarrow} \Map^h_{\Theta(\mathcal{M})}(\text{Can}^\pi X, \text{Can}^\pi Y)
\]
if $X,Y$ bifibrant.

Remark: The simplicial map of derived mapping spaces above arises from the functor
\[
(X \xrightarrow{\sim} W \leftarrow \sim Y) \longmapsto (\text{Can}^\pi X \longrightarrow \text{Can}^\pi W \leftarrow \sim \text{Can}^\pi Y)
\]
where we use that $\text{Can}^\pi$ is a left Quillen functor to deduce that $\text{Can}^\pi X$ is cofibrant (so that (**) applies) and that the map from $\text{Can}^\pi Y$ to $\text{Can}^\pi W$ is an acyclic cofibration.

Defn: The monad $\Theta$ satisfies effective homotopic descent if
\[
\text{Can}^\pi: \mathcal{M} \longrightarrow \Theta(\mathcal{M}) \text{ is a Quillen equivalence.}
\]

Remark: If $\Theta$ satisfies effective homotopic descent, then it also satisfies homotopic descent. Indeed,
\[
\text{Can}^\pi \text{ Quillen equivalence} \Rightarrow \tilde{\eta}_X: X \overset{\sim}{\longrightarrow} \text{Prim}^\pi(\text{Can}^\pi X), \forall X \text{ cofib.}
\]
so \( \exists \text{ diagram in } s\text{Set} \updownarrow X,Y \text{ bifibrant in } M \)

\[
\begin{align*}
\text{Map}^h (X,Y) \overset{\sim}{\longrightarrow} \text{Map}^h_M (X, \text{Prim}^\Pi (\text{Can}^\Pi X)^f) \\
\downarrow \sim M \uparrow \sim \quad \text{[Dwyer-H.]}
\end{align*}
\]

\[
\begin{align*}
\text{Map}^h_{\Theta(\Pi)} (\text{Can}^\Pi X, \text{Can}^\Pi Y) \overset{\sim}{\longrightarrow} \text{Map}^h_{\Theta(\Pi)} (\text{Can}^\Pi X, (\text{Can}^\Pi Y)^f).
\end{align*}
\]

**Questions:** How to determine when (effective) homotopic descent satisfied? What does it mean?

**Basic observations:** \( \Pi \) satisfies effective homotopic descent

\[
\begin{align*}
\text{(Using known characterization of Quillen equivalence)}
\end{align*}
\]

\[
\begin{align*}
\text{Realizability!}
\end{align*}
\]

\[
\begin{align*}
\text{Basic observations: } \Pi \text{ satisfies effective homotopic descent}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad \tilde{\eta}_X : X \overset{\sim}{\longrightarrow} \text{Prim}^\Pi (\text{Can}^\Pi X)^f \quad \text{if } X \text{ cofibrant} \\
\text{and}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad (A,m,\xi) \overset{f}{\longrightarrow} (A',m',\xi') \in \text{WE}_{\Theta(\Pi)} \iff \text{Prim}^\Pi f \in \text{WE}_{\Theta(\Pi)}
\end{align*}
\]

\[
\begin{align*}
\text{and}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad \varepsilon (A,m,\xi) : \text{Can}^\Pi (\text{Prim}^\Pi (A,m,\xi))^c \overset{\sim}{\longrightarrow} (A,m,\xi) \quad \text{if } f \text{ fibrant}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad X, X' \text{ cofibrant in } M \Rightarrow \text{rigidity problems!}
\end{align*}
\]

\[
\begin{align*}
\text{3) The descent spectral sequence: tool for studying homotopic descent, interpolating backwards along } \text{Map}^h_{\Theta(\Pi)} (X,Y) \overset{\sim}{\longrightarrow} \text{Map}^h_{\Theta(\Pi)} (\text{Can}^\Pi X, \text{Can}^\Pi Y)
\end{align*}
\]

Need more structure on \( M \) to do computations...

**Defn.** A simplicial model category consists of an \( s\text{Set} \)-enriched category \( M \) that is tensored and cotensored over \( s\text{Set} \) such that the underlying (ordinary) category \( M_0 \) is endowed with a model category structure, compatible with the enrichment in the sense that

\[
\begin{align*}
\text{(SMC): } i : A \overset{\longrightarrow}{\longrightarrow} X, \phi : E \longrightarrow B \Rightarrow (i^* \phi) : \text{Map} (X,E) \longrightarrow \text{Map} (A,E) \times \text{Map} (X,B)
\end{align*}
\]

\[
\begin{align*}
\text{(m.c.) } i : A \overset{\longrightarrow}{\longrightarrow} X, \phi : E \longrightarrow B \Rightarrow (\phi^* i) : \text{Map} (A,X) \longrightarrow \text{Map} (A,B)
\end{align*}
\]

\[
\begin{align*}
\text{Recall tensoring, cotensoring, etc.}
\end{align*}
\]
Rmk: If \( M \) is a simplicial model category, then
\[
\text{Map}^h_{M}(X,Y) \simeq \text{Map}^c(X^c,Y^c) \quad \forall X,Y \in \text{Ob}B.
\]

Defn: Let \( M \) be a simplicial model category. The totalization functor
\[
\text{Tot}: M^\Delta \rightarrow M
\]
(Need only cohesion for defn.)
is defined on objects by
\[
\text{Tot}(X^\bullet) = \text{equal } \left( \coprod_{n \geq 0} (X^n)^{\Delta[n]} \rightarrow \prod_{k,n \geq 0} (X^k)^{\Delta[n]} \right).
\]

Key properties:
1) \( X^\bullet \text{ Reedy fibrant} \Rightarrow \text{Tot} X^\bullet \text{ fibrant in } M. \)
2) \( \text{Tot}(cc^\bullet X) \simeq X \quad \forall X \in \text{Ob}M. \) (Exercise!)
3) \( cc^\bullet X \leftrightarrow Y^\bullet 2^\Delta \) "external" SDR in \( M^\Delta \)
\[
\Rightarrow X \simeq \text{Tot} Y^\bullet.
\]

Proof by
dualization of classical result
by J.-P. Meyer.

The descent SS associated to a monad \( \Pi \) is a special case of the following type of SS.

Defn: \([\text{Bousfield-Kan}]\) Let \( Y^\bullet \) be a Reedy fibrant cosimplicial simplicial object. The extended homotopy SS associated to \( Y^\bullet \) has
\[
E_{2}^{st} = \Pi^{s} \Pi^{t} Y^\bullet \quad \forall t \geq s \geq 0
\]

(in particular: \( t \geq 2 \Rightarrow E_{2}^{st} = H^\bullet(N_{*}(\Pi_{*} Y^\bullet)) \))
and abuts to (subtle convergence issues!)
\[
\Pi_{*}(\text{Tot} Y^\bullet).
\]

The cosimplicial simplicial set to which we apply the BKSS construction is defined as follows, for any monad on a simplicial model category \( M. \)

Rmk: \( M \) simplicial model cat \( \Rightarrow \) \( \text{H}(\Pi) \) simplicial in general (cohesion issues).
**Def**: Let \( Y \in \text{Ob} \, M \). The \( \mathcal{T} \)-cobar construction on \( Y \),
\[ \Omega^*_{\mathcal{T}} Y \in oM^A, \] is defined by
\[ \Omega^*_{\mathcal{T}} Y = (TY \overset{\mu}{\longrightarrow} T^2Y \overset{\mu}{\longrightarrow} T^3Y \overset{\mu}{\longrightarrow} \cdots) \]
(Exercise: Check that the cosimplicial identities hold.)

**Remark**: \( \exists \, \eta^* : ccY \longrightarrow \Omega^*_{\mathcal{T}} Y \) \( \forall \, Y \in M \)

If \( Y = U^*(Y,m) \), then \( \eta^* \) fits into an external SDR
\[ ccY \overset{m}{\longleftarrow} \Omega^*_{\mathcal{T}} Y \overset{p}{\longrightarrow} \]
and so \( Y = \text{Tot} \, \Omega^*_{\mathcal{T}} Y \). Also, \( \exists \) external SDR in \((M^\pi)^A\)

**Input data for the \( \mathcal{T} \)-descent SS**

\[ \begin{align*}
\circ \, f : X & \longrightarrow Y \in M, \text{ where } X \text{ cofibrant} \\
\circ \, j : \Omega^*_{\mathcal{T}} Y & \longrightarrow \hat{Y}^*, \text{ where } \hat{Y}^* \text{ is Reedy fibrant and levelwise } \mathcal{T} \text{-injective.}
\end{align*} \]

**Always levelwise \( \mathcal{T} \)-injective.**

**Lemma**: If \( U^* : \mathcal{C} \longrightarrow \mathcal{C} \) preserves acyclic cofibrations, then \( \hat{Y}^* \) exists \( \forall \, Y \in \text{Ob} \, M \).

**Def**: The \( \mathcal{T} \)-descent spectral sequence at \( f \) is the BKSS of the cosimplicial simplicial set \( \text{Map}_{oM}(X, \hat{Y}^*) \), pointed at \( j \circ \eta^* \circ cc^* f \)

**Notation**: \( E^q_f \)

**Interpretation**:

\[ (Ef)^q : \text{Map}_{oM}(X, \hat{Y}^*) \cong \text{Map}_{oM} \left( X, (\text{Prim}^\pi \text{Can}^\pi)^A \hat{Y}^* \right) \]

Using result of [Dwyer-Hess], or since \( X \) cofib, \( Y^A \) fib \( \forall \, n \)

So:
\[ (E^q_f)^{s,t} \cong \text{Map}_{\mathcal{D}(\mathcal{T})}^h (\text{Can}^\pi X, (\text{Can}^\pi)^A \hat{Y}^*) \].
Let's analyze the target component, assuming \( \Pi \) is simplicial.

\[ cc \colon \text{Can}^{\Pi Y} \to (\text{Can}^{\Pi})^\Delta \Omega^\ast_{\Pi Y} \to (\text{Can}^{\Pi})^\Delta \hat{Y} \quad \text{in } \mathscr{O}(\Pi)^\Delta \]

is a cosimplicial injective resolution, i.e.,

- \((\text{Can}^{\Pi})^\Delta \hat{Y} \) is Reedy fibrant and levelwise injective wrt to the monad associated to \( U_{\text{li}^\Pi} \colon \mathscr{O}(\Pi) \xrightarrow{\sim} M^{\Pi} \colon F_{\text{li}^\Pi} \), with underlying functor \( \psi \)

\[
(A, m, s) \mapsto (FA, \mu_A, T_A);
\]

- \( U_{\text{li}^\Pi} \text{Can}^{\Pi Y} = F^{\Pi Y} \simeq \text{Tot} (F^{\Pi})^\Delta \Omega^{\ast}_{\Pi Y} = \text{Tot} (U_{\text{li}^\Pi} \text{Can}^{\Pi})^\Delta \Omega^{\ast}_{\Pi Y} \)

and under reasonable hypotheses \( \simeq \) e.g., \( Y \) fibrant and \( \Pi \) preserves fibrant objects and \( \mathcal{E} \) eq. between fibrants.

So under good conditions

\[ F^{\Pi Y} \simeq \text{Tot} (F^{\Pi})^\Delta \hat{Y}, \]

justifying

\[ (E^\Pi)^{\text{set}} = \text{Ext}^{\text{set}}_{\mathscr{O}(\Pi)} (\text{Can}^{\Pi Y}, \text{Can}^{\Pi Y}). \]

\[ (E^\Pi)^{\infty} : \text{Tot Map}_{\mathcal{M}} (X, \hat{Y}) \simeq \text{Map}_{\mathcal{M}} (X, \text{Tot} \hat{Y}) \simeq \text{Map}_{\mathcal{M}} (X, \text{Prim}^{\Pi} (\text{Can}^{\Pi Y})^{f}) \]

\( \text{Tot} \hat{Y} \) can be seen as a \( \mathcal{T} \)-completion of \( Y \) in the following sense, which doesn't require the full strength of simplicial model categories.

- A morphism \( f \colon X \to Y \) in \( \mathcal{M} \) is a \( \mathcal{T} \)-equivalence if

\[ \text{Map}_{\mathcal{M}} (Y, \mathcal{U}^\ast (A, m)) \simeq \text{Map}_{\mathcal{M}} (X, \mathcal{U}^\ast (A, m)) \]

for all \((A, m) \in M^{\Pi}\) such that \((A, m)\) fibrant, i.e., \( A \) fibrant in \( \mathcal{M} \).

Exercise: Let \( f \colon X \to Y \in \mathcal{M} \), where \( X, Y \) cofibrant. Prove that

\[ \mathcal{T} f \in \mathcal{WE}_{\mathcal{M}} \Rightarrow f \text{ a } \mathcal{T} \text{-equivalence and that the converse holds if } \mathcal{M} \text{ is a simplicial model category.} \]
An object \( Z \) in \( M \) is \( \Pi \)-complete if
\[
f : X \to Y \ \Pi\text{-equivalence} \Rightarrow f^* : \text{Map}^\Pi_{\mathcal{D}}(Y, Z) \to \text{Map}^\Pi_{\mathcal{D}}(X, Z).
\]

**Exercise**: Show that:
(i) \( (A, m) \in \mathcal{D}^\Pi \Rightarrow A \ \Pi\text{-complete} \)
(ii) \( W \ \text{retract of } Z, Z \ \Pi\text{-complete} \Rightarrow W \ \Pi\text{-complete} \)
(iii) \( i) + ii) \Rightarrow \text{every } \Pi\text{-injective object is } \Pi\text{-complete} \)
(iv) \( Z, Z' \ \text{weakly equivalent} \Rightarrow \)
\( Z \ \Pi\text{-complete} \iff Z' \ \Pi\text{-complete} \).

**Lemma**: \( Z^\cdot \in \mathcal{D}^\Pi \) Reedy fibrant such that \( Z^n \ \Pi\text{-complete} \) then
\[\Rightarrow \text{Tot } Z^\cdot \ \Pi\text{-complete} \]
Here we assume that \( M \) is a simplicial model category.

Thus, under our hypotheses, \( \text{Tot } \hat{Y}^\cdot \) is indeed \( \Pi\text{-complete} \).

**Notation**: \( Y^\wedge = \text{Tot } \hat{Y}^\cdot \).

**Conclusion**: The \( \Pi\)-descent SS at \( f \) abouts to
\[\Pi^* \text{Map}_{\mathcal{D}^\Pi}(X, Y^\wedge) \approx \text{Map}_{\mathcal{D}^\Pi}(X, \text{Prim}^\Pi(\text{Can}^\Pi Y) f).\]

So the \( \Pi\)-descent SS can be seen as "interpolating backwards" from the target of \( \text{Map}^\Pi_{\mathcal{D}^\Pi}(X, Y) \to \text{Map}^\Pi_{\mathcal{D}^\Pi}(\text{Can}^\Pi X, \text{Can}^\Pi Y) \) to its source. It's therefore a tool for measuring deviation from satisfying homotopic descent.

**Example**: Let \( \varphi : A \to B \) be a monoid morphism in a monoidal model category such that \( \varphi(\varphi) \) admits a model category structure of the sort studied above.

Then \( \Omega^\cdot_{\varphi} \mathcal{M} \) is the well-known Amitsur complex.
If \( \Omega^\cdot_{\varphi} \mathcal{M} \) is Reedy fibrant, so that we may choose
completion of $M$ along $\mathfrak{F}$ (cf. e.g., [Carlsson]).

The $\mathfrak{F}$-descent SS looks like:

$$\text{Ext}_{\mathfrak{F}(q)}^{st}(\langle M@B, \rho_M \rangle, \langle N@B, \rho_N \rangle)$$

$$\Rightarrow \quad \pi_{t-s} \text{Map}_{\text{Mod}_A}(M, N@) \quad \text{indecomposables}.$$

Special cases:

1) $\eta: I \rightarrow B \Rightarrow$ Adams-type SS

2) $A \xrightarrow{\varepsilon} I \Rightarrow$ Quillen-homotopy SS

$$\text{Ext}_{\mathfrak{F}(q)}^{st}(M@Q, N@Q) \Rightarrow \pi_{t-s} \text{Map}_{\text{Mod}_A}(M, N@) \quad \text{htpy indecomposables}.$$

Remarks: 1) One should be able extract an "obstruction theory à la Bousfield" for measuring obstructions to realizing a $\mathfrak{F}$-alg as a free $\mathfrak{F}$-algebra and to realizing a morphism between free $\mathfrak{F}$-alg as an element of $\text{Im} F^\mathfrak{F}$ from the $\mathfrak{F}$-descent SS.

2) A dual theory of homotopic co-descent and an associated SS. Applied to the Ganea comonad, one should be able to use this machinery to study approximations to LS-category.

Example: Let $p: P@B \rightarrow B$ be a Kan fibration in $\text{sSet}$, where $P@B \simeq$. Consider $\phi: \text{sSet}/P@B \xrightarrow{\eta} \text{sSet}/B : \phi$. The canonical co-descent datum associated to $X \in P@B \xrightarrow{\phi@} \text{htpy@}(f) \rightarrow P@B$, where $\text{htpy}$ is the Kan loop gp on $B$. The associated comonad $\text{htpy}$ satisfies homotopic co-descent: $\text{Map}_{\text{sSet}/B}(f, q) \Rightarrow \text{Map}_{\text{sSet}(I\text{htpy})}(\text{htpy@}(f), \text{htpy@}(q)) \Rightarrow \text{htpy Kan fib}$. 

II. Homotopic Hopf-Galois extensions

A. Classical Hopf-Galois extensions [Chase-Sweedler], [Kreimer-Takeuchi]

Hopf-Galois data. Dualizing the notion of a Galois extension...

- $k$ commutative ring
- $H$ - a $k$-bialgebra
- $A$ - a $k$-algebra with trivial $H$-coaction
- $B$ - an $H$-comodule algebra with coaction $\rho : B \to B \otimes H$
- $\varphi : A \to B$ - a morphism of $H$-comodule algebras

Associated homomorphisms

- The Galois map
  \[ B \otimes_A B \xrightarrow{B \otimes_A \rho} B \otimes_A B \otimes H \xrightarrow{\tilde{\mu} \otimes H} B \otimes H \]
  \[ \beta \varphi \]

- The corestriction map
  \[ A \xrightarrow{\lambda \varphi} B^\co H = B \square_H k = \{ beB | \rho(b) = b \otimes 1 \} \]

Defn: $\varphi : A \to B^2 H$ is a Hopf-Galois extension if $\beta \varphi$ and $\lambda \varphi$ are both isomorphisms.

Examples: 1) Let $G$ be a finite group, $G < \text{Aut}_k(E)$ for some field extension $k \subset E$. Let $F = E^G$.

Then: $F \hookrightarrow E$ is a $G$-Galois extension

$\iff F \hookrightarrow E$ is a $k^G$-Hopf-Galois extension,

where $k^G = \text{Hom}_k(k, E)$. 

[Montgomery]
2) Let \( r : X \times G \to X \) be an action of a finite group on a finite set. Let \( Y = X_G = \) the set of \( G \)-orbits, and let \( q : X \to Y \) denote the quotient map.

Consider: \( X \times X \xrightarrow{\Delta \times G} X \times X \times X \times G \xrightarrow{\times r} X \times X \xrightarrow{\times r} X \times X \)

Let \( \mathbb{k} \) be a field. Let \( \mathbb{k}^G \) be defined as above, while \( \mathbb{k}^X = \text{Set} (X, \mathbb{k}) \), \( \mathbb{k}^Y = \text{Set} (Y, \mathbb{k}) \), endowed with pointwise + and \( \cdot \).

\[ \Rightarrow \text{Hopf-Galois data} \quad \mathbb{k}^Y \xrightarrow{q^*} \mathbb{k}^X \overset{\alpha}{\rightarrow} \mathbb{k}^G \]

**Exercise:** \( q^* \) is a \( \mathbb{k}^G \)-Hopf-Galois extension

\[ \iff \alpha \text{ is a bijection} \]

\[ \iff \text{the action } r \text{ is free.} \]

(Show that \( \beta_{q^*} = \alpha^* \).)

3) Let \( H \) be a \( \mathbb{k} \)-bialgebra, seen as an \( H \)-comodule algebra over itself. Consider \( q^H = \eta : \mathbb{k} \to H \).

Then: \( \eta \) is clearly an iso.

\[ H \otimes H \xrightarrow{\Delta \otimes H} H \otimes H \otimes H \xrightarrow{H \otimes \mu} H \otimes H \]

By \( \beta \), \( H \) is a Hopf algebra.

More generally, if \( H \) is a Hopf algebra, then

\[ \Delta_{\otimes H} : A \leftrightarrow A \otimes H \]

is an \( H \)-Hopf-Galois extension, of normal basis type.
Why algebraists care about HG-extensions
- Generalization of Galois theory
- Faithfully flat HG-extensions over coordinate ring of an affine group scheme correspond to G-principal fiber bundles
- Can study Hopf algebras via their associated HG-extensions

Why homotopy theorists might care

[Regnér] The unit map \( \eta: S \to \text{MU} \) is a homotopic HG-ext over \( S[\text{BU}] \), which is NOT a homotopic Galois extension for any group \( G \).

B. Homotopic HG-extensions

1) Co-rings and their comodules

Let \((V, \otimes, I)\) be a monoidal category. If \((A, \mu, \eta)\) is a monoid in \( V \), then \((A\text{-}\text{Mod}_A, \otimes_A, A)\) is also a monoidal category.

Defn: An \( A \)-co-ring is a comonoid in \( A\text{-}\text{Mod}_A \), i.e., \((V, S, \varepsilon)\) where:
- \( V \) is an \( A \)-bimodule,
- \( S: V \to V \otimes_A V \) coassociativity and
- \( \varepsilon: V \to A \) counitality.

Defn: Let \((V, S, \varepsilon)\) be a \( V \)-co-ring. An \textit{\( A \)-right \( V \)-comodule} consists of \((M, \rho)\) where \( M \in A\text{-}\text{Mod}_A \) and
\[
\rho: M \to M \otimes_A V \in A\text{-}\text{Mod}_A
\]
such that \((M \otimes_A S) \rho = (\rho \otimes_A V) \rho\), \((M \otimes_A \varepsilon) \rho = \text{Id}_M \).

Notation: \( V^A \) = category of \( A \)-right \( V \)-comodules in \( A\text{-}\text{Mod}_A \).
Remark: \( A = I \Rightarrow A \text{Mod}_A = V \) and an \( A \)-co-ring is just a
comonoid in \( V \).

\( V = A \Rightarrow V^A \cong \text{Mod}_A \).

Remark: If \((V, \delta, \epsilon)\) is an \( A \)-co-ring, then \((- \otimes_A V, - \otimes_A \delta, - \otimes_A \epsilon)\)

is a comonad on \( \text{Mod}_A \), so can apply \([BHKCRS]\) thm
to get model cat structure.

**Important adjunctions**

\( \text{The forgetful/cofree adjunction} \)

\[ V^A_A \xleftarrow{\Delta} \text{Mod}_A \xrightarrow{\epsilon} V^A \]

(Special case of the adjunction below, for \( g = \epsilon \))

\( \epsilon \) is the “restriction of coefficients” adjunction

\[ V^A_{V} \xrightarrow{g_*} V^W \]

where \( V(M, \rho) \in \text{Ob} \, V^W \),

\[ \rho \circ V = \text{equal}(M \otimes V_A \xrightarrow{\rho \circ V} M \otimes V' \otimes V) \]

\( -\text{computed in } V^A \).

**Homotopy theory**

Suppose now that \((M, \otimes, I)\) is a monoidal model category, which
is combinatorial and such that every object in \( M \) is small
relative to \( \text{Mor} \, M \).

Let \( A \) be a monoid in \( M \), and let \( V \) be an co-ring such that

\(- \otimes_A V \) preserves monos.

Let \( I \) denote the set of generating acyclic cofibrations of \( M \).
If \((f \otimes A) \text{-} \text{cell} \subseteq WE\), then by the Schwede-Shipley theorem

\[ \exists \text{Quillen pair } \mathcal{M} \xleftarrow{\perp} \text{Mod}_A \]

with \(\text{Fib}_{\text{Mod}_A} = U^*(\text{Fib}_A), \text{WE}_{\text{Mod}_A} = U^*(\text{WE}_A)\)

\[ \text{Cof}_{\text{Mod}_A} = (f \otimes A) \text{-} \text{cell}. \]

If \(\exists y \subseteq \text{Fib}_{\text{Mod}_A} \cap \text{WE}_{\text{Mod}_A} \) st \(\text{Fib}_{\text{Mod}_A} \cap \text{WE}_{\text{Mod}_A} = \text{Post}_y\), then by the HKKRS theorem:

\[ U(\text{Post}_y \otimes V) \subseteq WE \implies \exists \text{Quillen pair } U : V^V \xleftarrow{\perp} \text{Mod}_A : - \otimes V \]

with \(\text{WE}_{V^V} = U^*(\text{WE}_{\text{Mod}_A})\)

\[ \text{Cof}_{V^V} = U^*(\text{Cof}_{\text{Mod}_A}) \]

\[ \text{Fib}_{V^V} \cap \text{WE}_{V^V} = \text{Post}_y \otimes V. \]

Remark: Henceforth we always assume (*) holds for \(A\), and we consider only co-rings for which (**) holds.

Note that if (**) holds for \(V, W\), then any morphism of co-rings \(g : V \to W\) gives rise to a Quillen pair

\[ V^V \xleftarrow{\perp} W. \]

In particular, \(\exists \text{Quillen pair } \text{Mod}_A \xleftarrow{\perp} V^V\)

If coaugmented co-rings \(V\).

Example: \(M = \text{Ch}_{f \geq 0}^R\), \(A\) any dg \(R\)-algebra, \(V\) an \(A\)-co-ring that’s semi-free as a left \(A\)-module on a generating graded \(k\)-module of finite type

\(\implies\) have the desired model cat structure on \(V^V_A\).
2) Homotopify the $H^-$-framework

The data $\varphi: A \to B \otimes H$ that satisfies the monoid axiom:

- $H$ is a bimonoid with comultiplication $\Delta: H \to H \otimes H$
- $A$ is a monoid, seen as an $H$-comodule with trivial coaction $A \xrightarrow{\delta} A \otimes H$
- $B$ is an $H$-comodule monoid with coaction $p: B \to B \otimes H$, which is a monoid morphism
- $\varphi: A \to B$ is a morphism of $H$-comodule monoids.

Example: Continuation of the previous example ... -

Recall: $\Omega(M; C; N)$ where $C$ dg coalg, $M$ right $C$-comod, $N$ left $C$-comodule.

Let $H$ be a dg Hopf algebra, and let $E$ be a right $H$-comodule algebra.

Proposition: [H-Levi] The multiplication on $E$ extends naturally to $\Omega(E; H; \mathbb{k})$ and to $\Omega(E; H; H)$ so that the inclusion $\Omega(E; H; \mathbb{k}) \hookrightarrow \Omega(E; H; H)$ is $H^-$-data. 

- a homotopically normal basis extension

Associated co-rings:

- The canonical (or descent) co-ring: $W_{\varphi} = (B \otimes B, S_{\varphi}^1, e_{\varphi})$
  - already seen in very first lecture.

- The Hopf co-ring: $W_{\rho} = (B \otimes H, B \otimes \Delta, B \otimes e_{H})$, with $B$-actions $B \otimes W_{\rho} \xrightarrow{\delta_{\otimes H}} B \otimes H$

and $W_{\rho} \otimes B \xrightarrow{W_{\rho} \otimes p} W_{\rho} \otimes B \otimes H \xrightarrow{\sim} B \otimes B \otimes H \otimes H \xrightarrow{\Delta_{\otimes H}} B \otimes H$. 
The Hopf-Galois map \[ B \otimes_A B \xrightarrow{\beta^g} B \otimes H \]

is a morphism of \( B \)-co-rings. (Exercise!)

Introducing homotopy

Henceforth assume \( q^g: A \to B, H \) are such that \( E \) Quillen pairs

\[ M_B^{\text{Wq}} \xrightarrow{\perp} \text{Mod}_B \xleftarrow{\perp} M_B^{\text{Wp}} \]

and

\[ \text{Mon}^H \xleftarrow{\perp} \text{Mon} \]

wrt to model category structures of the type discussed above.

Consequently, \( E \) Quillen pair \( \text{Mon} \xleftarrow{\perp} \text{Mon}^H \).

Defn: A model for the homotopy coinvariants of a \( H \)-comod monoid \((B, \rho)\) is \( ((B, \rho)^f)^{\text{coH}} \) for some fibrant replacement of \((B, \rho)\) in \( \text{Alg}^H \). Notn: \((B, \rho)^{\text{coH}}\)

Example (cont.): \( \Omega(E; H; H) \) is a functorial fibrant replacement of \( E \) in \( \text{Alg}^H \) (cf. [H-Shipley]).

Consequently, \( E^{\text{coH}} = \Omega(E; H; H) \).

Defn: The homotopy corestriction map

\[ A \cong (A)^{\text{coH}} \xrightarrow{q^g} B^{\text{coH}} \xrightarrow{(B^f)^{\text{coH}}} \]

\[ \text{Lq} \]
Putting it all together...

**Defn:** \( \varphi : A \rightarrow B^2 \) is a homotopic \( H \)-Hopf-Galois extension if

\[
\begin{array}{c}
M^W_B \\ W_B^p \\
\end{array}
\xleftarrow{\mathbf{BG}^p} \quad \begin{array}{c}
\mathbf{M}^W_B \\ W_B^p \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\mathbf{Mod}^A \\ l_{\varphi}^* \\
\end{array}
\xrightarrow{\mathbf{BG}^p} \quad \begin{array}{c}
\mathbf{Mod}^B \\ B^2 \ker H \\
\end{array}
\]

are Quillen equivalences.

**Example (cont.)** \( \varphi : \Omega(E_j; H; \mathbb{k}) \rightarrow \Omega(E_j; H; H) \)

\( \bullet \) Since \( \Omega(E_j; H; H) \) is fibrant in \( \text{Alg}^H \), can take \( \Omega(E_j; H; H) \cong \Omega(E_j; H; \mathbb{k}) \), i.e., \( l_{\varphi} \) is an isomorphism.

\( \bullet \) \( \Omega(E_j; H; H) \otimes \Omega(E_j; H; \mathbb{k}) \xrightarrow{\mathbf{BG}^p} \Omega(E_j; H; H) \otimes \mathbb{k} \)

So \( \varphi \) is even a strict \( H \)-HG-extension and thus a homotopic HG-extension, since \( \Omega(E_j; H; H) \) is fibrant in \( \text{Alg}^H \).

**Example:** Let \( H \) be a simplicial monoid, seen as a simplicial bimonoid via the diagonal map.

Let \( A \) be a simplicial monoid.

Let \( B \) be a fibrant \( H \)-comodule monoid, i.e., a simplicial monoid endowed with a simplicial \( H \)-action \( \phi : B \rightarrow H \) that is also a Kan fibration.

Let \( \varphi : A \rightarrow B \) be a simplicial \( H \)-action \( \phi \), i.e., \( \varphi \) is a principal fibration.

**Remark:** If \( M = C_{\mathbb{k}} \), \( B \) is left \( A \)-semifree of finite type, and \( H \) is degreewise \( \mathbb{k} \)-free of finite type, then the desired model category structures exist on \( \text{Alg}^H \), \( M_B^W \), and \( M_B^W \), since \( W_B^p, W_B^p \) are then both \( B \)-semifree of finite type.
Properties of homotopic HG-extensions [Karpova]

Framework: $\mathcal{M} = Ch_{\simeq}^{>0}_{k}$

(Properties analogous to those proved by Rognes for homotopic Galois extensions of ring spectra.)

Theorem: Let $\varphi: A \to B_\varphi$ and $f: A \to A'$ be morphisms of commutative dg $k$-algebras such that $B$ and $A'$ are left $A$-semifree of finite type. In the pushout diagram

$$
\begin{array}{ccc}
A & \rightarrow & A' \\
\downarrow \varphi & & \downarrow \tilde{\varphi} \\
B & \rightarrow & A' \otimes_k B
\end{array}
$$

if $\varphi$ is a homotopic $H$: HG-extension, then so is $\tilde{\varphi}$.

Conversely...

Theorem: Under the same hypotheses as above, if $A'$ is faithfully flat over $A$, then:

$\varphi$ homotopic $H$: HG $\Rightarrow$ $\tilde{\varphi}$ homotopic $H$: HG.

Currently in progress: one direction of a Hopf-Galois correspondence, i.e., $\varphi: A \to B_\varphi$ homotopic HG-extension

$\Rightarrow$ (quotients of $H$ $\Rightarrow$ subextensions of $\varphi$).

Rmk: Under appropriate connectivity-nilpotency and finite-type conditions, any morphism of commutative dg algebras can be replaced up to quasi-isomorphism by a semifree extension of the type required in the theorems above. So, up to homotopy, the semifreeness hypothesis is not a real constraint.
From Grothendieck to Hopf and Galois via Koszul

Goal: To understand the big picture relating homotopic Grothendieck descent to homotopic Hopf-Galois extensions, at least for \( M = CH_{\mathbb{Q}} \), though the results stated here almost certainly hold more generally.

First we need one more notion: blending algebras and coalgebras.

A. Generalized Koszul duality

Motivation:

Let \( A \) and \( C \) be a dg \( \mathbb{k} \)-algebra and a dg \( \mathbb{k} \)-coalgebra, respectively. A twisting cochain from \( C \) to \( A \) is a \( \mathbb{k} \)-linear map

\[
t : C_* \rightarrow A_{*-1}
\]

of degree \(-1\) such that \( d_A t + t d_C = \mu(t \otimes t) \Delta \).

Key property: \( Alg(\Omega C, A) \leftrightarrow Tw(C, A) \leftrightarrow Coalg(C, BA) \)

\[
\alpha_t \leftrightarrow t \leftrightarrow R_t
\]

Theorem: [Lefèvre] If \( t : C \rightarrow A \) is a twisting cochain such that \( \alpha_t \) (equiv. \( R_t \)) is a quasi-isomorphism, then \( \exists \) Quillen equivalence

\[
-\otimes_C A : M^C \rightleftarrows M^A : -\otimes \mathbb{k} C \quad (!)
\]

In particular, if \( A \) is Koszul and \( C \) is its Koszul dual, then \( \exists \) Quillen equivalence \((!)

Example: Universal examples: \( M^C \rightleftarrows M^B_C \), \( M^{BA} \rightleftarrows M^A \).

Def: A coalgebra \( C \) is a generalized Koszul dual of \( A \) if \( \exists \) Quillen equivalence \( M^C \rightleftarrows M^A \).
B. Grothendieck vs Hopf-Galois

For any underlying monoidal (model) category, we have

\[
\begin{array}{ccc}
A \xrightarrow{\Phi} & B & \\
\uparrow & & \uparrow \\
A \xrightarrow{\Phi} & B^{\text{Gal}} & \\
\downarrow & & \downarrow \\
B^{\text{H}} & & \\
\end{array}
\xrightarrow{\text{Mon}}
\xrightarrow{\text{HopfGalois}}
\xrightarrow{\text{CoRing}}
\xrightarrow{\text{Hopf}}
\xrightarrow{\text{W_\Phi} = (B \otimes_A B, \delta_\Phi, \epsilon_\Phi)}
\xrightarrow{\text{W_\Phi} = (B \otimes H, B \otimes A, B \otimes e_A)}
\]

The following two theorems describe the precise relationship between homotopic Grothendieck descent and HG-extensions, at least when \( M = \text{Ch}^{20}_{\text{Ch}} \).

**Theorem I:** Let \( \Phi : A \xrightarrow{\Phi} B^{\text{Gal}} \) be Hopf-Galois data in \( \text{Ch}^{20}_{\text{Ch}} \).

If \( \Phi_0^* : M_B^{\text{ho}H} \rightarrow M_A \) is a Quillen equivalence, then:

\( \Phi \) homotopic HG-extension \( \iff \Phi \) satisfies effective homotopic Grothendieck descent.

**Theorem II:** Let \( \Phi : A \xrightarrow{\Phi} B^{\text{H}} \) be Hopf-Galois data in \( M = \text{Ch}^{20}_{\text{Ch}} \).

If \( \Phi_B : B \xrightarrow{\sim} B \), then

\( \Phi_0^* : M_B^{\text{ho}H} \rightarrow M_A \) is a Quillen equivalence

\( \iff \Phi \) is a generalized Koszul dual of \( A \).
Proof of the Theorems:

For any choice of Hopf-Galois data in $\text{Ch}_{k}^{\geq 0}$

$$A \xrightarrow{\varphi} B^{\mathcal{O}}$$

there is a commuting diagram of $HG$-data

$$\begin{array}{c}
\text{algebra morphism} \\
\downarrow \varphi \\
\text{morphism of } H\text{-comodule algebras}
\end{array}
$$

$$A \xrightarrow{\varphi} B^{\mathcal{O}}$$

The commuting diagram of $HG$-data above induces a commuting diagram of categories and functors. ($\mathcal{M} = \text{Ch}_{k}^{\geq 0}$)

\[
\begin{array}{c}
\text{Can}_\varphi \downarrow \complement \uparrow \\
\text{equivalence of categories}
\end{array}
\]

\[
\begin{array}{c}
\text{Can}_\varphi \downarrow \complement \uparrow \\
\text{isomorphism of categories}
\end{array}
\]

(\text{**}) Quillen equivalence, since $B \otimes H \xrightarrow{\sim} B' \otimes H$ as $B$-comings.

(\text{***}) No reason to expect $B \otimes_A B \rightarrow B' \otimes_{A'} B'$ to be a quasi-isomorphism in general.

(\text{****}) If $\mathfrak{k} \cong B$, then $\mathfrak{k} \cong B'$, and these are Quillen equivalences as in (\text{*}).
and it easily. The key to the proof is the particularly nice behavior of the "normal basis extension" $\varphi': A' \hookrightarrow B'$. 

Scholium: Given HG-data, $A \xrightarrow{\varphi} B^{\mathfrak{g}^H}$, if $A \subseteq B^c_0 H$, then $B \otimes_A B$ can be viewed as "generalized Koszul dual" of $A$. 