Homotopical algebra
Exercise Set 11

07.05.2018

Exercise 5 is to be handed in on 14.05.2018.

1. Let \((M, WE, Cof, Fib)\) be a model category, and let \(\gamma : M \to \text{Ho} M\) denote the canonical functor. Let \(g \in \text{Mor} M\). Show that \(\gamma(g)\) is an isomorphism if and only if \(g \in WE\).

2. Let \((M, WE_M, Cof_M, Fib_M)\) and \((N, WE_N, Cof_N, Fib_N)\) be model categories. Let \(F : M \to N\) and \(G : N \to M\) be functors such that \(F \dashv G\). Prove that the following conditions are equivalent.
   (a) \(F(Cof_M) \subseteq Cof_N\) and \(G(Fib_N) \subseteq Fib_M\)
   (b) \(F(Cof_M) \subseteq Cof_N\) and \(F(Cof_M \cap WE_M) \subseteq Cof_N \cap WE_N\)
   (c) \(G(Fib_N) \subseteq Fib_M\) and \(G(Fib_N \cap WE_N) \subseteq Fib_M \cap Fib_M\)

3. Prove Ken Brown’s Lemma:
   Let \((M, WE_M, Cof_M, Fib_M)\) and \((N, WE_N, Cof_N, Fib_N)\) be model categories, and let \(F : M \to N\) be a functor. If \(F\) sends every acyclic cofibration between cofibrant objects in \(M\) to a weak equivalence in \(N\), then \(F\) sends any weak equivalence between cofibrant objects to a weak equivalence.
   (Hint: Given an weak equivalence \(g : X \to Y\) between cofibrant objects in \(M\), consider the factorization of \(g + \text{Id}_Y : X \amalg Y \to Y\) as a cofibration followed by an acyclic fibration.)

4. Let \((M, WE, Cof, Fib)\) be a model category, and let \(F : M \to D\) be a functor. Suppose that \(F\) sends any acyclic cofibration between cofibrant objects to an isomorphism. Prove that that if \(f \sim g : X \to Y\) in \(M\), and \(X\) and \(Y\) are cofibrant, then \(F(f) = F(g)\).
   (Hint: Consider a very good right homotopy from \(f\) to \(g\).)

5. Let \((M, WE, Cof, Fib)\) be a model category. Combining Exercises 2 and 3, prove the following assertions.
(a) Let $J$ denote any of the small categories of Exercise set 9, Exercise 3. With respect to the model structures defined there, prove that if $\tau : \Phi \to \Psi$ is a weak equivalence between cofibrant objects in $M^J$, then $\operatorname{colim}_J \tau : \operatorname{colim}_J \Phi \to \operatorname{colim}_J \Psi$ is a weak equivalence in $M$.

(b) Let $J$ denote any of the small categories of Exercise set 10, Exercise 3. With respect to the model structures defined there, prove that if $\tau : \Phi \to \Psi$ is a weak equivalence between fibrant objects in $M^J$, then $\operatorname{lim}_J \tau : \operatorname{lim}_J \Phi \to \operatorname{lim}_J \Psi$ is a weak equivalence in $M$.

(c) Provide concrete examples to illustrate the necessity of the cofibrancy condition in (a) and of the fibrancy condition in (b).