Koszul duality and rational homotopy theory

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Plan of the presentation

- Motivation and some facts from rational homotopy theory
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- Motivation and some facts from rational homotopy theory
- Introduction to Koszul duality
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- Motivation and some facts from rational homotopy theory
- Introduction to Koszul duality
- Applications to rational homotopy theory
Convention

We assume that all spaces we consider are simply connected CW complexes.
Rational homotopy theory

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Definition

Let $f : X \to Y$ be a map of spaces, then we call $f$ a rational homotopy equivalence if one of the following equivalent conditions holds

1. $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism,
2. $f_* : H^*(X; \mathbb{Q}) \to H^*(Y; \mathbb{Q})$ is an isomorphism.
Theorem (Quillen)

There is an equivalence between the homotopy category of rational spaces and the homotopy category of differential graded Lie algebras over $\mathbb{Q}$. 
Theorem (Sullivan)

There is an equivalence between the homotopy category of rational spaces and the homotopy category of commutative differential graded algebras over $\mathbb{Q}$. 
Commutative differential graded algebras

**Definition**

A commutative differential graded algebra is a differential graded vector space $A$ together with a product

$$
\cdot : A \otimes A \rightarrow A.
$$

Such that $\cdot$ is

- associative
- graded commutative
  $$a \cdot b = (-1)^{\text{deg}(a)\text{deg}(b)} b \cdot a,$$
- and satisfies the Leibniz rule
  $$d(a \cdot b) = d(a) \cdot b + (-1)^{\text{deg}(a)} a \cdot d(b).$$
Quasi-free CDGA’s

Definition

A quasi-free CDGA \((\Lambda V, d)\), is a free commutative differential graded algebra \(\Lambda V\) together with a differential \(d\).
A Sullivan model for a space $X$ is a quasi-free commutative differential graded algebra

$$(\wedge V, d) \xrightarrow{\sim} A_{PL}(X),$$

where $A_{PL}(X)$ is the algebra of polynomial de Rham forms. We also require that the differential satisfies some properties.
Sullivan models

**Definition**

A Sullivan model for a space $X$ is a quasi-free commutative differential graded algebra

$$(\wedge V, d) \sim \rightarrow A_{PL}(X),$$

where $A_{PL}(X)$ is the algebra of polynomial de Rham forms. We also require that the differential satisfies some properties.

**Definition**

A Sullivan model $(\wedge (V), d)$ is called minimal if $d(V) \subseteq \wedge^{\geq 2} V$. 
The minimal Sullivan model is unique up to isomorphism and two spaces $X$ and $Y$ are rational homotopy equivalent if and only if their minimal Sullivan models are isomorphic.
Minimal Sullivan models

Theorem

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Theorem

Let $X$ be a space and $(\wedge V, d)$ its minimal Sullivan model, then there is an isomorphism

$$\pi_k(X) \otimes \mathbb{Q} \cong \text{Hom}_\mathbb{Q}(V^k, \mathbb{Q}).$$
Examples of Sullivan models

Example (Odd dimensional spheres)
The minimal Sullivan model for $S^{2n+1}$ is given by $\Lambda a$ the free CDGA on one generator $a$ of degree $2n + 1$ and with a zero differential.

Example (Even dimensional spheres)
The minimal Sullivan model for $S^{2n}$ is given by $\Lambda a, b$ such that $\deg(a) = 2n$ and $\deg(b) = 4n - 1$ and the differential is given by $d(b) = a^2$. 
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Question

How do we construct a minimal Sullivan model for an algebra $A$?
Question

*How do we construct a minimal Sullivan model for an algebra A?*

Answer

*Some complicated inductive algorithm.*
But in special cases we have a technique which makes computing the minimal Sullivan model extremely easy.
Koszul duality

Convention

For simplicity we will now work with associative algebras. But all these ideas work in much greater generality.
In general if we want a free resolution of an algebra $A$ we have the following standard resolution which is given by

$$\Omega BA \sim A.$$ 

Where $\Omega$ is the cobar construction and $B$ the bar construction.
The bar construction

Definition

The bar construction is a functor

\[ B : \text{Associative algebras} \rightarrow \text{Coassociative coalgebras} \]

which is given by \( BA = (T^c(sA), d_B) \).
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Definition

The cobar construction is a functor

\[ \Omega : \text{Coassociative coalgebras} \rightarrow \text{Associative algebras} \]

which is given by \( \Omega C = (T(s^{-1}C), d_\Omega) \).
The bar and cobar construction

**Proposition**

*The bar and cobar construction both preserve quasi-isomorphisms.*
In the case that the coalgebra $BA$ is formal, i.e. $BA$ is quasi isomorphic to its homology $HBA$, then $\Omega HBA$ is also a resolution for $A$ and it is minimal.
Definition

An algebra $A$ is called Koszul if the bar construction is formal.
Koszul duality

Definition
An algebra $A$ is called Koszul if the bar construction is formal.

Theorem
If $A$ is a Koszul algebra then $\Omega HBA$ is a minimal model for $A$. 
Quadratic algebras

Definition

An algebra $A$ is called quadratic if it has a presentation of the form

$$A = T(V)/(R),$$

where $R \subseteq V \otimes V$. 
Homology of the bar construction

**Definition**

Let $A$ be a quadratic algebra, then we define an extra grading, called the weight grading, on $A$ by giving each generator degree 1.
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**Definition**

On the bar construction of $A$ we define an extra grading called the bar length, which is defined by assigning every element in $A$ degree 1.
The bar complex

<table>
<thead>
<tr>
<th>Bar length</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$V$</td>
<td>$d$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{V^2}{R}$</td>
<td>$d$</td>
<td>$V \otimes^2$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{V^3}{(VR \oplus RV)}$</td>
<td>$d$</td>
<td>$(\frac{V^2}{R} \otimes V) \oplus (V \otimes \frac{V^2}{R})$</td>
</tr>
</tbody>
</table>
The diagonal $D$ of the bar construction is the sub complex of the bar construction of all elements such that the bar length is equal to the weight.
The diagonal $\mathcal{D}$ of the bar construction is the sub complex of the bar construction of all elements such that the bar length is equal to the weight.

Note that the homology of the diagonal is very easy to compute.
Definition (Alternative definition of Koszul duality)

An algebra $A$ is Koszul if $HBA \subseteq D$
The Koszul dual

**Definition**

Let $A$ be a quadratic algebra then we define the Koszul dual algebra $A^!$ as

$$A^! = T(sV^*)/(R^\perp),$$

In this case $R^\perp$ is the annihilator of $R$ under the pairing $V^* \otimes V^* \otimes V \otimes V \to \mathbb{Q}$. 
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If $A$ is Koszul then $A^!$ is isomorphic to the $(HBA)^*$. 
Example

The algebra $\mathbb{Q}[x]/x^2$ is Koszul.
Examples

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Koszul duality can be generalized to many situations, a few examples are

- Operads
- Algebras over operads
- Theorem
  - The operads $\text{COM}$ and $\text{LIE}$ are Koszul dual to each other.
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- Operads

Koszul duality and rational homotopy theory
Generalizations of Koszul duality

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- Operads
- Algebras over operads

**Theorem**

*The operads $\mathbb{COM}$ and $\mathbb{LIE}$ are Koszul dual to each other.*
A space $X$ is called a Koszul space if it is rationally equivalent to the derived spatial realization of a Koszul algebra.
Examples of Koszul spaces

- Spheres
Examples of Koszul spaces

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- Loop spaces
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- Products and wedges of Koszul spaces
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- Ordered configuration spaces of points in $\mathbb{R}^n$
- Highly connected manifolds
Applications

Theorem (Berglund)

Let $X$ be a Koszul space then the homotopy and cohomology groups are Koszul dual, i.e. we have an isomorphism

$$\pi_*(\Omega X) \otimes \mathbb{Q} = H^*(X; \mathbb{Q})^{\text{Lie}}.$$
Applications

Theorem (Berglund)

Let $X$ be an $n$-connected Koszul space then there is the following isomorphism

$$H_*(\Omega^n X; \mathcal{Q}) \cong H^* (X; \mathcal{Q})^{!G_n}.$$
Example: The wedge of two spheres

- The space $S^2 \vee S^3$ is a Koszul space.
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- The space $S^2 \vee S^3$ is a Koszul space.
- The cohomology is given by $\mathbb{Q}[x, y]/(x^2, xy, y^2)$ with $\text{deg}(x) = 2$ and $\text{deg}(y) = 3$. 

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Example: The wedge of two spheres

- The space $S^2 \vee S^3$ is a Koszul space.
- The cohomology is given by $\mathbb{Q}[x, y]/(x^2, xy, y^2)$ with $\text{deg}(x) = 2$ and $\text{deg}(y) = 3$.
- The rational homotopy Lie algebra of the loop space is then given by $\pi_*(\Omega(S^2 \vee S^3)) \otimes \mathbb{Q} = LIE(a, b)$, with $\text{deg}(a) = 1$ and $\text{deg}(b) = 2$. 
Thank you for your attention.