A visual introduction to cyclic sets and cyclotomic spectra

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July 7, 2015
Young Topologists Meeting
Lausanne, Switzerland
Goal: the cyclic bar construction and topological Hochschild homology ($THH$) in pictures.

Key idea: “cyclotomic” structure.
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Key idea: “cyclotomic” structure.

Useful for algebraic $K$-theory. And fun!
G a monoid, X a right G-space, Y a left G-space.
$G$ a monoid, $X$ a right $G$-space, $Y$ a left $G$-space.

$$B(X, G, Y) = \left| [k] \mapsto X \times G^\times k \times Y \right|$$

\[
\begin{array}{c}
X \\
\downarrow d_0 \\
X \times G \\
\downarrow d_1 \\
X \times G \times G \\
\downarrow d_2 \\
X \times G \times G \times G \\
\downarrow d_3 \\
X \times G \times G \times G \times Y
\end{array}
\]
$G$ a monoid, $X$ a right $G$-space, $Y$ a left $G$-space.

$$B(X, G, Y) = |[k] \mapsto X \times G^\times k \times Y|$$

$X \times G \times G \times G \times Y$

$d_0$ $d_1$ $d_2$ $d_3$

Recipe for a space: one $\Delta^k$ for each $(x, g_1, \ldots, g_k, y)$. 
G a monoid, X a right G-space, Y a left G-space.

\[ B(X, G, Y) = \left\{ [k] \mapsto X \times G^\times k \times Y \right\} \]

\[ X \times G \times G \times G \times G \times Y \]

\[ d_0 \quad d_1 \quad d_2 \quad d_3 \]

Recipe for a space: one \( \Delta^k \) for each \( (x, g_1, \ldots, g_k, y) \).

\[ EG = B(\ast, G, G), \quad BG = B(\ast, G, \ast) \]
The cyclic bar construction.

The circle action and fixed points.

Elementary examples.

Cyclic spectra and \( THH \).

Review of the bar construction.

Also works for:

- based spaces with smash product
- abelian groups with tensor product
- spectra with the smash product
- diagrams ("\( G \) has many objects")
The cyclic bar construction. 

The circle action and fixed points. 

Elementary examples. 

Cyclic spectra and $\text{THH}$. 

The cyclic bar construction.

$$B^\text{cyc} G = [[k] \mapsto G \times G^\times k]$$
The terms $G^{\times k+1}$ form a cyclic space.
The terms $G \times ^{k+1}$ form a *cyclic* space.

\[ \Delta^{\text{op}} \subseteq \Lambda^{\text{op}} \longrightarrow \text{Top} \]

- $\Delta^{\text{op}}$: totally ordered sets
- $\Lambda^{\text{op}}$: "cyclically ordered sets"
The terms $G \times k^{+1}$ form a \textit{cyclic} space.

\[
\Delta^{\text{op}} \text{ (totally ordered sets)} \subset \Lambda^{\text{op}} \text{ ("cyclically ordered sets")} \longrightarrow \text{Top}
\]

\[
\text{ob} \Delta = [0], [1], [2], [3], \ldots
\]
The terms $G \times k + 1$ form a *cyclic* space.

\[
\Delta^\text{op} \quad \subset \quad \Lambda^\text{op} \quad \longrightarrow \quad \text{Top}
\]

\[
\text{totally ordered sets} \quad \subset \quad \text{“cyclically ordered sets”}
\]

\[
\text{ob} \Delta = \quad , \quad , \quad , \quad , \quad , \quad \ldots
\]

\[
\text{ob} \Lambda = \quad , \quad , \quad , \quad , \quad , \quad \ldots
\]

The morphisms are “degree 1” functors.
Here’s a morphism $f : [2] \to [8]$ in $\Lambda$
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It sends $G^9$ to $G^3$ like this:

$$G \times G^8 \to G \times G^2$$
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It sends $G^9$ to $G^3$ like this:

$$G \times G^8 \to G \times G^2$$

$$g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \mapsto g_6 g_7 g_8 g_0, g_1 g_2 g_3 g_4 g_5, 1$$
To make *Topological Hochschild homology*, just form $B^{\text{cyc}}$ in the category of spectra.
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$R$ a ring spectrum.
Theorem

If $X_\bullet : \Lambda^{\text{op}} \to \text{Top}$ is a cyclic space, the realization $|X_\bullet|$ has a natural action by the circle group $S^1$. 
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Proof: $X_\bullet$ always a colimit of cyclic sets $\Lambda(-, [n])$ for varying $n$. 

The cyclic bar construction. 

The circle action and fixed points. 

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The circle action. 

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Theorem

If $X_\bullet : \Lambda^{\text{op}} \to \text{Top}$ is a cyclic space, the realization $|X_\bullet|$ has a natural action by the circle group $S^1$.

Proof: $X_\bullet$ always a colimit of cyclic sets $\Lambda(-, [n])$ for varying $n$.

Just need the circle action on $\Lambda^n := |\Lambda(-, [n])|$.
Simplices in $\Lambda^n \leftrightarrow$ maps $[k] \rightarrow [n]$. 
Simplices in $\Lambda^n \leftrightarrow$ maps $[k] \rightarrow [n]$. Lift to the “universal cover” of $[n]$: 

![Diagram showing simplices in $\Lambda^n$ and their maps to $[n]$.]
Simplices in $\Lambda^n \leftrightarrow$ maps $[k] \rightarrow [n]$.
Lift to the “universal cover” of $[n]:$

\[ f : \{0, \ldots, k\} \rightarrow \{(0, 0), (0, 1), \ldots, (0, n), (1, 0), (1, 1), \ldots, (1, n)\}. \]
Simplices in $\Lambda^n \leftrightarrow$ maps $[k] \longrightarrow [n]$.
Lift to the “universal cover” of $[n]$:

$\leftrightarrow$ an increasing function

$f : \{0, \ldots, k\} \longrightarrow \{(0, 0), (0, 1), \ldots, (0, n), (1, 0), (1, 1), \ldots, (1, n)\}$.

Unique, unless $f(k) \leq (0, n)$ or $f(0) \geq (1, 0)$. 
Glue the top to the bottom: $\Lambda^0 \cong \Delta^0 \times S^1$
The cyclic bar construction.

The circle action and fixed points.

Elementary examples.

Cyclic spectra and $THH$.

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Glue the top to the bottom: $\Lambda^1 \cong \Delta^1 \times S^1$
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Glue the top to the bottom: $\Lambda^2 \cong \Delta^2 \times S^1$
The cyclic bar construction.

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Glue the top to the bottom: $\Lambda^3 \cong \Delta^3 \times S^1$
Glue the top to the bottom: $\Lambda^3 \cong \Delta^3 \times S^1$ and so on. □
$C_n \leq S^1$ cyclic subgroup — what are its fixed points?
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Simplicial level 0: get one copy of $\Lambda^0$ for each $g \in G$
$C_n \leq S^1$ cyclic subgroup — what are its fixed points?

Simplicial level 0: get one copy of $\Lambda^0$ for each $g \in G$

Degenerate if $g = 1$, nondegenerate otherwise.
Simplicial level 1: we get a $\Lambda^1 = \Delta^1 \times S^1$ for each pair $(g_1, g_2)$.
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The bottom triangle for $(g_1, g_2)$ is glued to top triangle for $(g_2, g_1)$ and vice-versa.
Simplicial level 1: we get a $\Lambda^1 = \Delta^1 \times S^1$ for each pair $(g_1, g_2)$.

The bottom triangle for $(g_1, g_2)$ is glued to top triangle for $(g_2, g_1)$ and vice-versa.
Are any blue points fixed by some nontrivial element of $S^1$?
Answer: only the midpoint, and only if $g_1 = g_2$:
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The given point must hit itself on the red line again, and only the midpoint does this.
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The given point must hit itself on the red line again, and only the midpoint does this.
We get a $G \times \Lambda^0$ in the $C_2$-fixed points.
Simplicial level 2: play the same game. One prism for each triple \((g_1, g_2, g_3)\)
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\[
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{2}
\end{array}
\]

\[
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{2}
\end{array}
\]

glued by rotating the triple \((g_1, g_2, g_3)\) and rotating the three 3-simplices in the figure.
Simplicial level 2: play the same game. One prism for each triple \((g_1, g_2, g_3)\)

![Diagram](attachment:image.png)

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![Diagram]

glued by rotating the triple $(g_1, g_2, g_3)$ and rotating the three 3-simplices in the figure.
Which points in the blue simplex are fixed?
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Which points in the blue simplex are fixed? Triple must be \((g_1, g_1, g_1)\), point must be fixed under rotation of vertices of \(\Delta^2\) \(\leadsto\) only the barycenter.
We get \(C_3\)-fixed points:

Get another \(G \times \Lambda^0\) in the \(C_3\)-fixed points.
Simplicial level 3: look for fixed points in $G^4 \times \Delta^3$. 
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We get a $G \times \Lambda^0$ in the $C_4$-fixed points.
Next chance to get mapped to yourself, by $C_2$:
Next chance to get mapped to yourself, by \( C_2 \):

\[
\begin{align*}
(0, t_1, t_2, t_3) &\mapsto (t_2, t_3, t_0, t_1) \\
0 &\rightarrow 1 \\
1 &\rightarrow 2 \\
2 &\rightarrow 0 \\
3 &\rightarrow 0
\end{align*}
\]

More fixed points! \( C_2 \) acts on \( \Delta^3 \) by rotating the coordinates twice:

The fixed points form a line \( \Delta^1 \).
So, get a copy of $G^2 \times \Lambda^1$ in the $C_2$-fixed points.
So, get a copy of $G^2 \times \Lambda^1$ in the $C_2$-fixed points.

Can easily formalize now: if $r \mid n$, the piece $G^n \times \Lambda^{n-1}$ has $C_r$-fixed points $G^{n/r} \times \Lambda^{n/r-1}$. 
Collect it all together:

<table>
<thead>
<tr>
<th>simp. level</th>
<th>$S^1$</th>
<th>$C_1$</th>
<th>$C_2$</th>
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<tr>
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Notice anything?

\[
(B_{cyc}^c G)^{C_n} \cong (B_{cyc}^c G)^{C_1} = B_{cyc}^c G
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Notice anything?

\[(B^{\text{cyc}} G)^{C_n} \cong (B^{\text{cyc}} G)^{C_1} = B^{\text{cyc}} G\]

An $S^1$-space with this property is *cyclotomic*. 
$X$ any unbased space, the free loop space is $LX = \text{Map}(S^1, X)$.
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$C_n$-fixed loops must follow the same path $n$ times:

$$(LX)^{C_n} \simeq LX$$
$X$ any unbased space, the free loop space is $LX = \text{Map}(S^1, X)$.

$C_n$-fixed loops must follow the same path $n$ times:

$$(LX)^{C_n} \cong LX$$

In fact

**Proposition**

$$B^{\text{cyc}} G \cong L(BG)$$
Switch to based spaces and smash product.
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$X$ a based space, $S^0 \vee X$ the “square zero extension” of $S^0$ by $X$. 
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**Proposition**

$B_{\text{cyc}}^c(S^0 \vee X) \cong S^0 \vee (\Lambda^0/\partial \wedge X) \vee (\Lambda^1/\partial \wedge_{C_2} X \wedge X) \vee (\Lambda^2/\partial \wedge_{C_3} X^3) \vee \ldots$
Switch to based spaces and smash product. $X$ a based space, $S^0 \vee X$ the “square zero extension” of $S^0$ by $X$.

**Proposition**

\[
B_{\text{cyc}}(S^0 \vee X) \cong \\
S^0 \vee (\Lambda^0 / \partial \wedge X) \vee (\Lambda^1 / \partial \wedge C_2 X \wedge X) \vee (\Lambda^2 / \partial \wedge C_3 X \wedge X^3) \vee \ldots
\]

\[
T(X) = S^0 \vee X \vee X^2 \vee X^3 \vee \ldots
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**Proposition**

$B^{\text{cyc}}(S^0 \vee X) \cong S^0 \vee (\Lambda^0/\partial \wedge X) \vee (\Lambda^1/\partial \wedge_c X \wedge X) \vee (\Lambda^2/\partial \wedge_c X^0) \vee \ldots$

$T(X) = S^0 \vee X \vee X^2 \vee X^3 \vee \ldots$

**Proposition**

$B^{\text{cyc}}(T(X)) \cong S^0 \vee (\Lambda^0_+ \wedge X) \vee (\Lambda^1_+ \wedge_c X \wedge X) \vee (\Lambda^2_+ \wedge_c X^3) \vee \ldots$
Switch to based spaces and smash product.
$X$ a based space, $S^0 \lor X$ the “square zero extension” of $S^0$ by $X$.

**Proposition**

$B_{\text{cyc}}(S^0 \lor X) \cong S^0 \lor (\Lambda^0 / \partial \land X) \lor (\Lambda^1 / \partial \land C_2 X \land X) \lor (\Lambda^2 / \partial \land C_3 X^3) \lor \ldots$

$T(X) = S^0 \lor X \lor X^2 \lor X^3 \lor \ldots$

**Proposition**

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Very similar!
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**Proposition**

$B^{\text{cyc}}(S^0 \vee X) \cong S^0 \vee (\Lambda^0 / \partial \wedge X) \vee (\Lambda^1 / \partial \wedge C_2 X \wedge X) \vee (\Lambda^2 / \partial \wedge C_3 X^3) \vee \ldots$

$T(X) = S^0 \vee X \vee X^2 \vee X^3 \vee \ldots$

**Proposition**

$B^{\text{cyc}}(T(X)) \cong S^0 \vee (\Lambda_+^0 \wedge X) \vee (\Lambda_+^1 \wedge C_2 X \wedge X) \vee (\Lambda_+^2 \wedge C_3 X^3) \vee \ldots$

Very similar! (Koszul duality)
Apply $B^{\text{cyc}}$ to a ring spectrum $R$, result is $\text{THH}(R)$. 
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$$\Phi^C_n THH(R) \cong THH(R)$$
Apply $B^{cyc}$ to a ring spectrum $R$, result is $THH(R)$. Above arguments apply verbatim, if we use orthogonal spectra and geometric fixed points:

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Earlier model (Bökstedt): extra coherence machinery
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$$\Phi^n THH(R) \cong THH(R)$$

Earlier model (Bökstedt): extra coherence machinery
Applications: $THH(DX)$ and its dual, mapping spectra between cyclotomic spectra, bivariant algebraic $K$-theory.
Takeaway: $THH$ is cool!

The face maps of the cyclic bar construction, superimposed on the objects of $\Lambda$. 