$n$-Butterflies: Modeling Derived Morphisms of Strict $n$-Groups

Gregory (Ivan) Dungan II

Department of Mathematics
USMA, West Point

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A homotopy $n$-type is an object $X$ of $Ho(\text{Top}_\mathbb{Q})$ in which 
$\pi_k(X) = 1$ for $k > n$. 
$n$-Homotopy Types

- A **homotopy $n$-type** is an object $X$ of $\text{Ho}(\text{Top}_Q)$ in which $\pi_k(X) = 1$ for $k > n$.
- The category of homotopy $n$-types is the full subcategory

\[ \text{H}_n\text{Typ} \subseteq \text{Ho}(\text{Top}_Q). \]
A **homotopy $n$-type** is an object $X$ of $Ho(\text{Top}_Q)$ in which $\pi_k(X) = 1$ for $k > n$.

The category of homotopy $n$-types is the full subcategory $H_n\text{Typ} \subseteq Ho(\text{Top}_Q)$.

Moreover, $H_1\text{Typ} \subseteq H_2\text{Typ} \subseteq H_n\text{Typ} \subseteq Ho(\text{Top}_Q)$. 
The functor $\pi_1 : \text{Top}^c \to \text{Grp}$ induces $H^1\text{Typ}^c \simeq \text{Grp}$
Connected Homotopy 1-Types

- The functor $\pi_1 : \text{Top}^c \to \text{Grp}$ induces
  \[ H^1\text{Typ}^c \simeq \text{Grp} \]

- Groups model connected homotopy 1-types.
The functor $\pi_1 : \text{Top}^c \to \text{Grp}$ induces $\text{H1Typ}^c \simeq \text{Grp}$.

- Groups model connected homotopy 1-types.
- $[X, Y]_{\text{Top}} \simeq \text{Grp}(\pi_1(X), \pi_1(Y))$ where $X, Y$ are connected homotopy 1-types.
Crossed Modules

A crossed module \([G : \partial]\) is a homomorphism of groups 
\(\partial : C_2 \rightarrow C_1\) with a right action \(x^a\) of \(G_1\) on \(G_2\) satisfying

\[
\begin{align*}
\text{CM1} & \quad \partial(x^a) = a^{-1}\partial(x)a \\
\text{CM2} & \quad x^{\partial(y)} = y^{-1}xy
\end{align*}
\]
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\]

A morphism \(f : [G, \partial] \to [H, \delta]\) is a commutative diagram

\[
\begin{array}{ccc}
G_2 & \xrightarrow{f_1} & H_2 \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{f_1} & H_1
\end{array}
\]

such that \(f_2\) is \(f_1\)-equivariant.
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  \]

  such that \(f_2\) is \(f_1\)-equivariant.

- Crossed modules with morphisms form a category \(\text{xm}\).
Connected Homotopy 2-Types

Theorem (B. Noohi [?])

The Moerdijk-Svensson model structure on $\text{xm}$ induces the equivalence

$$\text{H}_2\text{Typ}^c \simeq \text{Ho}(\text{xm})$$

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The Moerdijk-Svensson model structure on $\mathbf{xm}$ induces the equivalence

$$\mathbf{H2Typ}^c \simeq \mathbf{Ho(xm)}$$

- Crossed modules model connected homotopy 2-types.
Connected Homotopy 2-Types

Theorem (B. Noohi [?])
The Moerdijk-Svensson model structure on $\mathbf{xm}$ induces the equivalence

$$H^2\text{Typ}^c \simeq Ho(\mathbf{xm})$$

- Crossed modules model connected homotopy 2-types.
- The morphisms $[X, Y]_{\mathbf{xm}}$ model morphisms of connected homotopy 2-types.
Morphisms of $\text{H}^2\text{Typ}$

- $[H, G]_{xm} = \text{xm}(Q, G) \simeq$ where $Q$ is a cofibrant replacement of $H$. 

Theorem (B. Noohi) 

There is a bijection $[H, G]_{xm} \sim \pi_0(B(H, G))$ where $B(H, G)$ is the groupoid of butterflies.

The connected components of $B(G, H)$ model morphisms of connected homotopy $2$-types.
Morphisms of $\mathbf{H}_2\mathbf{Typ}$

- $[H, G]_{xm} = \text{xm}(Q, G) / \simeq$ where $Q$ is a cofibrant replacement of $H$.
- We would like to avoid computing cofibrant replacements.
Morphisms of $\text{H}^2\text{Typ}$

- $[H, G]_{\text{xm}} = \text{xm}(Q, G)/ \cong$ where $Q$ is a cofibrant replacement of $H$.
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**Theorem (B. Noohi [?])**

There is a bijection

$$[H, G]_{\text{xm}} \xrightarrow{\cong} \pi_0(B(H, G))$$

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- $[H, G]_{xm} = \text{xm}(Q, G) / \simeq$ where $Q$ is a cofibrant replacement of $H$.
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**Theorem (B. Noohi [?])**

There is a bijection

$$[H, G]_{xm} \overset{\simeq}{\rightarrow} \pi_0(B(H, G))$$

where $B(H, G)$ is the groupoid of *butterflies*.

- The connected components of $B(G, H)$ model morphisms of connected homotopy 2-types.
A butterfly from \([G : \partial]\) to \([H : \delta]\) is a commutative diagram

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\alpha} & H_2 \\
\downarrow{\partial} & & \downarrow{\delta} \\
E & \xrightarrow{p} & G_1 \\
\downarrow{f} & & \downarrow{\beta} \\
H_1 & & \end{array}
\]

where both diagonals are complexes, \(H_2 \to E \to G_1\) is short exact and for \(x \in E, g \in G_2, h \in H_2\)

\[
\alpha(g^p(x)) = x^{-1}\alpha(g)x \quad \beta(h^f(x)) = x^{-1}\beta(h)x
\]
A morphism of butterflies is an isomorphism $\Theta : E \to E'$ such that

\[
\begin{array}{ccc}
G_2 & \to & E \\
\downarrow & & \Theta \\
G_1 & \to & E'
\end{array}
\]

commutes.
A morphism of butterflies is an isomorphism $\Theta : E \to E'$ such that

\[ \begin{array}{c}
G_2 \\
\downarrow \\
G_1
\end{array} \quad \begin{array}{c}
E \\
\downarrow \\
E'
\end{array} \quad \begin{array}{c}
H_2 \\
\downarrow \\
H_1
\end{array} \]

commutes.

Butterflies from $[G : \partial]$ to $[H : \delta]$ with morphisms form a groupoid denoted by $B(G,H)$. 
Question

▶ Can we model morphisms of other spaces up to homotopy type?
Can we model morphisms of other spaces up to homotopy type?

In particular, is there an analog of butterflies for these spaces?
A crossed complex $[G, \delta]$ over a groupoid $G_1$ is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} G_k \xrightarrow{\delta_k} G_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} G_2 \xrightarrow{\delta_2} G_1 \xrightarrow{\delta_0} G_0$$
A crossed complex \([G, \delta]\) over a groupoid \(G_1\) is a sequence

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\]

1. For \(k \geq 2\), \(G_k = \{G_k(x)\}_{x \in G_0}\) where \(G_k(x)\) is a group.
A crossed complex \([G, \delta]\) over a groupoid \(G_1\) is a sequence

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1. For \(k \geq 2\), \(G_k = \{G_k(x)\}_{x \in G_0}\) where \(G_k(x)\) is a group.
2. For \(k \geq 3\), \(G_k(x)\) is abelian.
A crossed complex \([G, \delta]\) over a groupoid \(G_1\) is a sequence

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2. For \(k \geq 3\), \(G_k(x)\) is abelian.
3. \(\delta\) is a functor which respects \(G_0\) such that \(\delta \circ \delta = 1\).
A crossed complex \([G, \delta]\) over a groupoid \(G_1\) is a sequence

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4. \(G_1\) acts on \(G_k\) on the right and satisfies:
   4.1 For \(a \in G_k(x), f \in G_1(x, y)\), then \(af \in G_k(y)\).
   4.2 For \(k \geq 2\), \(\delta_k\) preserves the action.
   4.3 \(\text{Im}\delta_2\) acts by conjugation on \(G_2\) and trivially on \(G_k\) for \(k > 2\).
Crossed Complexes over a Groupoid

- A crossed complex \( [G, \delta] \) over a groupoid \( G_1 \) is a sequence

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\]

1. For \( k \geq 2 \), \( G_k = \{ G_k(x) \}_{x \in G_0} \) where \( G_k(x) \) is a group.
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   4.2 For \( k \geq 2 \), \( \delta_k \) preserves the action.
   4.3 \( \text{Im} \delta_2 \) acts by conjugation on \( G_2 \) and trivially on \( G_k \) for \( k > 2 \).

- \( [G, \delta] \) is a reduced crossed complex if \( G_1 \) is a group.
Categories of Crossed Complexes

▶ A morphism $f : H \to G$ is the data:

1. Set map $f_0 : H_0 \to G_0$
2. Functors $f^k : H^k \to G^k$ over $f_0$ compatible with $\delta$ and the action.

▶ Crossed complexes with morphisms form a category $\text{Xc}$. 
▶ Reduced crossed complexes form a full subcategory $\text{xc}$. 
▶ Reduced $n$-Crossed Complexes $\text{xc}^n$: $G^k = 1$ for all $k > n$. 

Gregory (Ivan) Dungan II  
Young Topologist Meeting 2015
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Crossed complexes with morphisms form a category $\mathbf{Xc}$.

Reduced crossed complexes form a full subcategory $\mathbf{xc}$.

Reduced $n$-Crossed Complexes $n\mathbf{xc} : G_k = 1$ for all $k > n$
Truncated Examples

- $\mathbf{xc}^1 : G_k = 0$ for $k \geq 1$

$$\mathbf{xc}^1 \simeq \text{Grp} \simeq \text{H}^1\text{Typ}^c$$
Truncated Examples

- \( \text{xc}^1 : G_k = 0 \) for \( k \geq 1 \)

\[
\text{xc}^1 \simeq \text{Grp} \simeq H_1 \text{Typ}^c
\]

- \( \text{xc}^2 : G_k = 0 \) for \( k \geq 2 \)

\[
\text{xc}^2 \simeq \text{xm} \leadsto H_0(\text{xc}^2) \simeq H_2 \text{Typ}^c
\]
Examples

- Initial object in $Xc$ is the empty crossed complex $\emptyset$
Examples

- Initial object in $\mathbf{Xc}$ is the empty crossed complex $\emptyset$
- Final object in $\mathbf{Xc}$

1: \[ \cdots \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \]
Examples

- Initial object in $Xc$ is the empty crossed complex $\emptyset$
- Final object in $Xc$

\[
\begin{array}{c}
1 : \quad \cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \\
\end{array}
\]
Examples

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\[
1 : \cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1
\]
Examples

- Initial object in $\mathbf{Xc}$ is the empty crossed complex $\emptyset$
- Final object in $\mathbf{Xc}$

  $1 : \cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1$

- The *unit interval crossed complex* $I$:
  1. $I_0 = \{0, 1\}$
Examples

- Initial object in \( \mathbf{Xc} \) is the empty crossed complex \( \emptyset \)

- Final object in \( \mathbf{Xc} \)

  \[
  \begin{CD}
  1 : @>>> 1 @>>> 1 @>>> 1 @>>> 1 @>>> 1
  \end{CD}
  \]

- The *unit interval crossed complex* \( I \):
  1. \( I_0 = \{0, 1\} \)
  2. \( I_k(0) = 1 \) and \( I_k(1) = 1 \)
Examples

- Initial object in $\mathbf{Xc}$ is the empty crossed complex $\emptyset$
- Final object in $\mathbf{Xc}$

$$
1 : \quad \cdots \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1
$$

- The unit interval crossed complex $I$:
  1. $I_0 = \{0, 1\}$
  2. $I_k(0) = 1$ and $I_k(1) = 1$
  3. $I_1(0, 1) = <i>$ and $I_1(1, 0) = <i^{-1}>$
The Moore complex gives an extension of the Dold-Kan correspondence to the category of reduced crossed complexes $\text{xc}$.

\[
s\text{AbGrp} \xrightarrow{\text{Norm}(-)} \text{Ch}_{\geq 0}(\mathbb{Z}) \\
\text{Grp}^T \text{Cmplx} \xrightarrow{\text{Moore}(-)} \text{xc}
\]
A \textit{m-fold left homotopy} \( (g, \phi^m_k : H_k \rightarrow G_{k+m}) : [H : \partial] \rightarrow [G : \delta] \) is a morphism

\[
\cdots \rightarrow G_{m+2} \rightarrow G_{m+1} \rightarrow \cdots \rightarrow G_2 \rightarrow G_1 \\
\uparrow g_{m+2} \quad \quad \uparrow g_{m+1} \\
\cdots \rightarrow H_{m+2} \rightarrow H_{m+1} \rightarrow \cdots \rightarrow H_2 \rightarrow H_1
\]
A \textit{m-fold left homotopy} \((g, \phi^m_k : H_k \to G_{k+m}) : \left[H : \partial\right] \to \left[G : \delta\right]\) is a morphism

\begin{align*}
\cdots & \to G_{m+2} & \leftrightarrow & \cdots & \to G_{m+1} & \leftrightarrow & \cdots & \to G_2 & \to G_1 \\
\uparrow{g_{m+2}} & & \uparrow{g_{m+1}} & \uparrow{\phi^m_2} & \uparrow{\phi^m_1} & \uparrow{g_2} & \uparrow{g_1} \\
\cdots & \to H_{m+2} & \to H_{m+1} & \to \cdots & \to H_2 & \to H_1
\end{align*}
A \textit{m-fold left homotopy} \((g, \phi^m_k : H_k \to G_{k+m}) : [H : \partial] \to [G : \delta]\) is a morphism

\[
\cdots \to G_{m+2} \leftarrow G_{m+1} \leftarrow \cdots \to G_2 \to G_1 \\
\cdots \to H_{m+2} \to H_{m+1} \to \cdots \to H_2 \to H_1
\]

1. \(\phi^m_1(ab) = \phi^m_1(a)g_1^m(b)\phi^m_1(b)\) for \(a, b \in H_1\);
**$k$-Fold Left Homotopy**

A *$m$-fold left homotopy* $\left( g, \phi_k^m : H_k \to G_{k+m} \right) : [H : \partial] \to [G : \delta]$ is a morphism

\[
\cdots \to G_{m+2} \leftarrow \cdots \to G_{m+1} \leftarrow \cdots \to G_2 \to G_1 \\
\cdots \to H_{m+2} \to H_{m+1} \to \cdots \to H_2 \to H_1 \\
\]

1. $\phi_1^m(ab) = \phi_1^m(a)g_1(b)\phi_1^m(b)$ for $a, b \in H_1$;
2. $\phi_k^m(xy) = \phi_k^m(x)\phi_k^m(y)$ for $x, y \in H_k$ where $k \geq 2$;
A \textit{m-fold left homotopy} $(g, \phi^m_k : H_k \to G_{k+m}) : [H : \partial] \to [G : \delta]$ is a morphism:

\[
\cdots \longrightarrow G_{m+2} \xleftarrow{g_{m+2}} G_{m+1} \xleftarrow{g_{m+1}} \cdots \longrightarrow G_2 \longrightarrow G_1 \\
\cdots \longrightarrow H_{m+2} \longrightarrow H_{m+1} \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1
\]

1. \(\phi^m_1(ab) = \phi^m_1(a)g_1(b)\phi^m_1(b)\) for \(a, b \in H_1\);
2. \(\phi^m_k(xy) = \phi^m_k(x)\phi^m_k(y)\) for \(x, y \in H_k\) where \(k \geq 2\);
3. \(\phi^m_k\) is \(g_1\)-equivariant for \(k \geq 2\).
Internal Hom and Tensor

Theorem (R. Brown, P. Higgins [?])

For crossed complexes $H, G$, there is a crossed $\mathbf{XC}(H, G)$ given by

$\mathbf{XC}(H, G)_0 = Xc(H, G)$

$\mathbf{XC}(H, G)_k = \{k – \text{fold left homotopies}\}$
Internal Hom and Tensor

Theorem (R. Brown, P. Higgins [?])

For crossed complexes $H, G$, there is a crossed $\mathbf{XC}(H, G)$ given by

$\mathbf{XC}(H, G)_0 = \mathbf{Xc}(H, G)$

$\mathbf{XC}(H, G)_k = \{k - \text{fold left homotopies}\}$

Theorem (R. Brown, P. Higgins [?])

For every $C, D, E \in \mathbf{Xc}$,

$\mathbf{Xc}(C \otimes D, E) \cong \mathbf{Xc}(C, \mathbf{XC}(D, E))$

which makes $(\mathbf{Xc}, \otimes, 1)$ a closed symmetric monoidal category.
$n$-Homotopy Groups

For a crossed complex $[G: \delta]$ and $x \in G_0$

- Connected Components: $\pi_0(G) = \pi_0(G_1)$;
\(n\)-Homotopy Groups

For a crossed complex \([G : \delta]\) and \(x \in G_0\)

- Connected Components: \(\pi_0(G) = \pi_0(G_1)\);
- Fundamental Homotopy Group: \(\pi_1(G, x) = \text{coker}\delta_2(x)\);
$n$-Homotopy Groups

For a crossed complex $[G : \delta]$ and $x \in G_0$

- Connected Components: $\pi_0(G) = \pi_0(G_1)$;
- Fundamental Homotopy Group: $\pi_1(G, x) = \text{coker} \delta_2(x)$;
- $n$-Homotopy Group: $\pi_n(G, x) = H_n(G(x))$. 
Weak Equivalences and Fibrations

A weak equivalence is a morphism $f : H \rightarrow G$ in $\mathbf{Xc}$ which induces

\[
\pi_0(H) \cong \pi_0(G) \\
\pi_k(H, x) \cong \pi_k(G, f_0(x))
\]
Weak Equivalences and Fibrations

- A **weak equivalence** is a morphism $f : H \to G$ in $\mathbf{Xc}$ which induces

  \[ \pi_0(H) \cong \pi_0(G) \]

  \[ \pi_k(H, x) \cong \pi_k(G, f_0(x)) \]

- A **fibration** is a morphism $f : H \to G$ in $\mathbf{Xc}$ such that
  
  1. $f_1 : H_1 \to G_1$ is a fibration of groupoids;
Weak Equivalences and Fibrations

A weak equivalence is a morphism $f : H \to G$ in $\mathbf{Xc}$ which induces
\[
\pi_0(H) \cong \pi_0(G), \quad \pi_k(H, x) \cong \pi_k(G, f_0(x))
\]

A fibration is a morphism $f : H \to G$ in $\mathbf{Xc}$ such that
1. $f_1 : H_1 \to G_1$ is a fibration of groupoids;
2. $f(x)_k : H_k(x) \to G_k(f_0(x))$ is a surjection for all $x \in H_0$ and $k \geq 2$. 
Weak equivalences and fibrations form a closed model structure on $\mathbb{X}_c$. 

Theorem (R. Brown, M. Golasinski [?])
Weak equivalences and fibrations form a closed model structure on $\mathbf{Xc}$.

- The homotopy category of $\mathbf{Xc}$ has as morphisms

$$[H, G]_{\mathbf{Xc}} = \mathbf{Xc}(Q, G)/\simeq$$

where $Q$ is a cofibrant replacement of $H$. 
Homotopy

For \( f, g : H \rightarrow G \), a homotopy from \( f \) to \( g \) is a morphism

\[
h : H \otimes I \rightarrow G
\]

such that

\[
\begin{array}{c}
H \\
i_0 \\
\downarrow \\
H \otimes I \\
\downarrow \\
H \\
i_1 \\
\end{array}
\begin{array}{c}
f \\
\downarrow \\
h \\
\downarrow \\
g \\
\end{array}
\begin{array}{c}
I \\
\rightarrow \\
G
\end{array}
\]

commutes.
Relation to 1-Fold Left Homotopy

Theorem (A. Tonks [?])

Let $f, g : H \rightarrow G$ be morphisms of reduced crossed complexes. Defining a homotopy $h : f \simeq g$ is equivalent to defining a 1-fold left homotopy

$$(g, \phi_k : H_k \rightarrow G_{k+1})$$
Relation to 1-Fold Left Homotopy

Theorem (A. Tonks [?])

Let \( f, g : H \to G \) be morphisms of reduced crossed complexes. Defining a homotopy \( h : f \simeq g \) is equivalent to defining a 1-fold left homotopy

\[
(g, \phi_k : H_k \to G_{k+1})
\]

Moreover, \( f \) is determined by

\[
\begin{align*}
    f_1(a) &= g_1(a)\delta_2(\phi_1(a)) \\
    f_k(x) &= g_k(x)\delta_{k+1}(\phi_k(x))\phi_{k-1}(\partial_k(x))
\end{align*}
\]
Relation to 1-Fold Left Homotopy

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\end{align*}
\]

- In other words, the quotient set \([H, G]_{\mathcal{X}c} = \mathcal{X}c(Q, G) / \simeq\) can be described using 1-fold left homotopies.
We would like to model \([H, G]_{xc} = Xc(Q, G)/\sim\).
Definition

- We would like to model $[H, G]_{x_c} = Xc(Q, G)/\sim$.
- Define *derived morphisms* to be the elements of the set $Xc(Q, G)$. 
We would like to model $[H, G]_{xc} = Xc(Q, G)/\simeq$.

Define *derived morphisms* to be the elements of the set $Xc(Q, G)$.

Derived morphisms can be viewed as fractions:

\[
\begin{array}{c}
Q \\
\downarrow p \\
\simeq \\
\downarrow f \\
H \quad \quad \rightarrow \quad \rightarrow \quad G
\end{array}
\]

where $p : Q \to H$ is a cofibrant replacement of $H$. 
The derived groupoid $\text{Rhom}(H, G)$ is defined by $\text{Rhom}(H, G)_0 = Xc(Q, G)$ and morphisms of the form...
The derived groupoid $\text{Rhom}(H, G)$ is defined by $\text{Rhom}(H, G)_0 = Xc(Q, G)$ and morphisms of the form $
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The derived groupoid $\text{Rhom}(H, G)$ is defined by $\text{Rhom}(H, G)_0 = Xc(Q, G)$ and morphisms of the form

$$\begin{array}{ccc}
Q & \xrightarrow{\phi} & G \\
\downarrow & & \downarrow \\
g & & f
\end{array}$$

where $\phi$ is a 1-fold left homotopy.

By definition, there is a bijection

$$[H, G]_{xc} \cong \pi_0(\text{Rhom}(H, G)).$$
Model of Derived Morphisms

- The main result:

\[ \text{Theorem (D.)} \]

Let \( H, G \) be reduced \( n \)-crossed complexes. Then there is an equivalence of categories

\[ \text{Rhom}(H, G) \simeq nB(H, G) \]

where \( nB(H, G) \) is the groupoid of \( n \)-butterflies.

\[ \text{Corollary} \]

Let \( H, G \) be reduced \( n \)-crossed complexes. Then there is a bijection

\[ [H, G]_{xc} \sim \pi_0(nB(H, G)) \]

where \( nB(H, G) \) is the groupoid of \( n \)-butterflies.
Model of Derived Morphisms

- The main result:

**Theorem (D.)**

Let $H, G$ be reduced $n$-crossed complexes. Then there is an equivalence of categories

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Model of Derived Morphisms

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**Corollary**

Let $H, G$ be reduced $n$-crossed complexes. Then there is a bijection

$$[H, G]_{xc} \cong \pi_0(nB(H, G))$$

where $nB(H, G)$ is the groupoid of $n$-butterflies.
Algebraic Replacement

- Goal: avoid computing a cofibrant replacement of $H$. 
Algebraic Replacement

- Goal: avoid computing a cofibrant replacement of $H$.
- Instead, find a crossed complex $E$ which satisfies

\[
\begin{array}{ccc}
Q & \xrightarrow{p} & E \\
\downarrow & & \downarrow f \\
H & \sim & G
\end{array}
\]
Factorization

For a derived morphism $f : Q \rightarrow G$, consider the morphism

$\nabla^f : Q \rightarrow H \times G$. 

But not necessarily a fraction!
For a derived morphism $f : Q \to G$, consider the morphism

$$\nabla^f : Q \to H \times G.$$ 

Then the following diagram commutes.

\[
\begin{array}{ccc}
Q & \xrightarrow{\nabla^f} & H \times G \\
\downarrow p & & \downarrow f \\
H & \xleftarrow{\pi_1} & G \\
\end{array}
\]

But not necessarily a fraction!
For a derived morphism $f : Q \to G$, consider the morphism

$$\nabla^f : Q \to H \times G.$$ 

Then the following diagram commutes.

But not necessarily a fraction!
For a derived morphism $f : [Q : \xi] \to [G : \delta]$ in $n\text{xc}$, there is a reduced $n$-crossed complex

$$H_n \times G_n \to Q_{n-1} \times \nabla^f_n H_n \times G_n \to Q_{n-2} \to Q_{n-3} \to \cdots$$
For a derived morphism \( f : [Q : \xi] \to [G : \delta] \) in \( nxc \), there is a reduced \( n \)-crossed complex

\[
H_n \times G_n \xrightarrow{} Q_{n-1} \times \nabla^f_n H_n \times G_n \xrightarrow{} Q_{n-2} \xrightarrow{} Q_{n-3} \xrightarrow{} \cdots
\]

where in the \( n = 2 \) case replace product with semidirect product.
For a derived morphism $f : [Q : \xi] \to [G : \delta]$ in $n\text{xc}$, there is a reduced $n$-crossed complex

$$H_n \times G_n \longrightarrow Q_{n-1} \times \nabla^f_n H_n \times G_n \longrightarrow Q_{n-2} \longrightarrow Q_{n-3} \longrightarrow \cdots$$

where in the $n = 2$ case replace product with semidirect product.

We will call this crossed complex the $n$-pushout below $\nabla^f_n$ and denote it by $[Q^f : \xi^f]$. 
$n$-Pushout below $\nabla^f_n$

- Let $f : [H : \partial] \to [G : \delta]$ be a morphism $nxc$ and $Q$ a cofibrant replacement of $H$. Then we have the factorization:

$$
\begin{array}{ccc}
Q & \xrightarrow{\iota} & Q^f \\
\downarrow & \downarrow & \downarrow \\
\nabla^f & \Rightarrow & H \times G
\end{array}
$$

\[\text{Theorem (D.)} \]

The morphism $\iota : Q \to Q^f$ is a weak equivalence.

\[\text{Theorem (D.)} \]

The morphism $\text{cotr}_{n-1}(\iota) : \text{cotr}_{n-1}(Q) \to \text{cotr}_{n-1}(Q^f)$ is an isomorphism in degree $n-1$ and the identity for $k < n-1$. 
**Results**

**n-Pushout below** \( \nabla_n^f \)

- Let \( f : [H : \partial] \to [G : \delta] \) be a morphism \( n \times c \) and \( Q \) a cofibrant replacement of \( H \). Then we have the factorization:

\[
\begin{align*}
\nabla f \\
\downarrow \downarrow \\
Q & \xrightarrow{\iota} Q^f & H \times G
\end{align*}
\]

**Theorem (D.)**

The morphism \( \iota : Q \to Q^f \) is a weak equivalence.
\( n \)-Pushout below \( \nabla^f_n \)

- Let \( f : [H : \partial] \to [G : \delta] \) be a morphism \( n \times c \) and \( Q \) a cofibrant replacement of \( H \). Then we have the factorization:

\[
\begin{align*}
\nabla^f_n & \quad \downarrow \quad \downarrow \\
Q & \quad \iota \quad Q^f \quad \rho \quad H \times G
\end{align*}
\]

**Theorem (D.)**

The morphism \( \iota : Q \to Q^f \) is a weak equivalence.

**Theorem (D.)**

The morphism \( \text{cotr}_{n-1}(\iota) : \text{cotr}_{n-1}(Q) \to \text{cotr}_{n-1}(Q^f) \) is an isomorphism in degree \( n - 1 \) and the identity for \( k < n - 1 \).
Induced Fraction

- The following diagram commutes.

\[
\begin{array}{ccc}
Q & \cong & Q^f \\
\downarrow \alpha & & \downarrow f \\
H \times G & \cong & H \times G \\
\downarrow \rho & & \downarrow \pi_1 \\
H & & G \\
\end{array}
\]
Induced Fraction

- The following diagram commutes.

\[
\begin{array}{c}
\text{Q} \\
\downarrow^\sim \\
\text{Q}^f \\
\downarrow^\rho \\
\text{H} \times \text{G} \\
\downarrow^\pi_1 \\
\text{H} \\
\downarrow^\omega \\
\text{G}
\end{array}
\quad \text{f}
\]

\[
\begin{array}{c}
\text{H} \\
\downarrow^\pi_2 \\
\text{G}
\end{array}
\]

Theorem (D.)

The induced morphism \( Q^f \rightarrow G \times H \xrightarrow{\pi_1} G \) is a trivial fibration.
The following diagram commutes.

\[
\begin{array}{ccc}
Q & \rightarrow & \Pi \times G \\
\downarrow & \searrow & \downarrow \\
Q' & \rightarrow & H \\
\end{array}
\]

Theorem (D.)

The induced morphism \( Q' \rightarrow \Pi \times H \xrightarrow{\pi_1} G \) is a trivial fibration.
By unfolding the map $Q^f \xrightarrow{\rho} H \times G$, we have a commutative diagram:

$$
\begin{array}{cccc}
H_n & \xrightarrow{} & Q_{n-1} \times \nabla^f_n (G_n \times H_n) & \xrightarrow{} & G_n \\
\downarrow & & \downarrow & & \downarrow \\
H_{n-1} & \xrightarrow{} & Q_{\leq n-2} & \xrightarrow{} & G_{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
H_{\leq n-2} & \xrightarrow{} & & \xrightarrow{} & G_{\leq n-2}
\end{array}
$$
A \textit{n-Butterfly} from $H$ to $G$ is

\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc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Definition Continued

- the induced sequences

\[ 1 \rightarrow G_n \xrightarrow{\beta} E_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times \ker \partial_{n-2} H_{n-1} \rightarrow 1 \]

\[ E_k \xrightarrow{u_k} \ker \eta_{k-1} \times \ker \partial_{k-1} H_k \rightarrow 1 \]

for \( k \leq n - 2 \) are exact;
Definition Continued

- the induced sequences

\[ 1 \to G_n \xrightarrow{\beta} E_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times \ker \partial_{n-2} \to H_{n-1} \to 1 \]

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for \( k \leq n - 2 \) are exact;

- the compositions \( \eta_{n-1} \circ (\alpha \times \beta) \) and \( f_n \circ \alpha \) are complexes
Definition Continued

- the induced sequences

\[
1 \longrightarrow G_n \xrightarrow{\beta} E_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times \ker \partial_{n-2} H_{n-1} \longrightarrow 1
\]

\[
E_k \xrightarrow{u_k} \ker \eta_{k-1} \times \ker \partial_{k-1} H_k \longrightarrow 1
\]

for \( k \leq n - 2 \) are exact;

- the compositions \( \eta_{n-1} \circ (\alpha \times \beta) \) and \( f_n \circ \alpha \) are complexes

- \( \alpha, \beta \) satisfy the compatibility conditions

\[
\alpha \left(x^{p_1(a)}\right) = \alpha(x)^a \quad \text{and} \quad \beta \left(y^{f_1(a)}\right) = \beta(y)^a
\]
Folding a $n$-Butterfly

**Theorem (D.)**

Let $([E, \eta], p, f, \alpha, \beta)$ be a $n$-butterfly from $G$ to $H$. Then the induced morphism

$$
\begin{array}{c}
H_n \times G_n \overset{\pi_1}{\longrightarrow} H_n \\
\downarrow \alpha \times \beta \downarrow \downarrow \partial_n \\
E_{\leq n-1} \overset{p}{\longrightarrow} H_{\leq n-1}
\end{array}
$$

of reduced $n$-crossed complexes is a trivial fibration.
Folding a $n$-Butterfly

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\downarrow \alpha \times \beta \quad \quad \quad \quad \downarrow \partial_n \\
E_{\leq n-1} \xrightarrow{p} H_{\leq n-1}
\end{array}$$

of reduced $n$-crossed complexes is a trivial fibration.

- We denote the folded $n$-butterfly on the left by $E^*$. 
Corollary

Let $p : Q \to H$ be a cofibrant replacement of $H$. Then there exists a lift $l$.

\[\begin{array}{ccc}
Q & \xrightarrow{\sim} & H \\
\downarrow & & \downarrow \\
\sim & & \sim \\
\end{array}\]

\[\begin{array}{ccc}
E^* & & \\
\downarrow & & \downarrow \\
\sim & & \sim \\
\end{array}\]
Corollary

Let $p : Q \to H$ be a cofibrant replacement of $H$. Then there exists a lift $l$ such that

\[
\begin{array}{ccc}
E^* & \xrightarrow{l} & Q \\
\downarrow & & \downarrow \sim \\
H & \xrightarrow{p} & H
\end{array}
\]

Definition

Let $Q$ be a cofibrant replacement of $H$. A \textit{n-butteryfly over Q} is an $n$-butterfly with a lift $l$ such that $\text{cotr}_{n-1}(l) : \text{cotr}_{n-1}(Q) \to \text{cotr}_{n-1}(E^*)$ is an isomorphism in degree $n - 1$ and the identity for $k < n - 1$. 
Morphisms of $n$-Butterflies

A morphism of $n$-butterflies over $\mathbb{Q}$ from $H$ to $G$ is a diagram

$$
\begin{array}{ccccccccc}
H_n & \rightarrow & E_{n-1} & \rightarrow & G_n \\
\downarrow & & \downarrow & \Theta_{n-1} & \downarrow \\
H_{n-1} & \rightarrow & E_{\leq n-2} & \rightarrow & G_{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
H_{\leq n-2} & \rightarrow & & E'_{\leq n-2} & \rightarrow \ G_{\leq n-2} \\
\end{array}
$$
\( n \)-Butterflies Groupoid

- Where \( \Theta \) is an isomorphism in degree \( n - 1 \), the identity for \( k < n - 1 \), and makes the diagram commute up to a left 1-fold homotopy \( \phi \).

\[
\begin{array}{ccc}
E & \xrightarrow{f} & G \\
\downarrow{\Theta} & & \downarrow{\phi} \\
E' & \xleftarrow{f'} & G
\end{array}
\]
$n$-Butterflies Groupoid

- where $\Theta$ is an isomorphism in degree $n - 1$, the identity for $k < n - 1$, and makes the diagram commute up to a left 1-fold homotopy $\phi$.

Theorem (D.)
The $n$-butterflies from $H$ to $G$ over $Q$ with the morphisms form a groupoid denoted by $nB(H, G)$. 
Property of Morphisms of $n$-Butterflies

Corollary

Let $(\mathcal{H}, \phi) : ([E, \eta], p, f, \alpha, \beta) \to ([E', \eta'], p', f', \alpha', \beta')$ be a morphism of $n$-butterflies. Then the induced morphism $E^* \to (E')^*$ of reduced $n$-crossed complexes is a weak equivalence.
Corollary

Let \((\Theta, \phi) : ([E, \eta], p, f, \alpha, \beta) \to ([E', \eta'], p', f', \alpha', \beta')\) be a morphism of \(n\)-butterflies. Then the induced morphism \(E^* \to (E')^*\) of reduced \(n\)-crossed complexes is a weak equivalence.

Theorem (D.)

Let \(H, G\) be reduced \(n\)-crossed complexes. Then there is an equivalence of categories

\[
\text{Rhom}(H, G) \cong nB(H, G).
\]

Moreover, there is a bijection

\[
[H, G]_{xc} \cong \pi_0(nB(H, G)).
\]
Thank you. Questions?