On the Alexander Polynomial of a welded ribbon tangle

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(joint work with Vincent Florens)
Panoramic view

Ribbon tangles in $B^4$

Alexander functor: functorial generalization of the Alexander polynomial

Welded tangle diagrams in $B^2$

Alexander polynomial: combinatorial invariant

Classical tangles: 2012, Bigelow, Cattabriga, Florens.


Ribbon tubes: 2015, Audoux, Bellingeri, Meilhan, Wagner.
Panoramic view

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Alexander functor: functorial generalization of the Alexander polynomial


Alexander polynomial: combinatorial invariant

- \( B^4 = B^3 \times [0, 1] \);
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- \( k_+ \) circles in the upper copy of \( B^3 \), and \( k_- \) circles in the lower copy of \( B^3 \);

**Diagram:**

- \( k_+ \) disjoint, unlinked, oriented, trivially embedded circles
- \( k_- \) disjoint, unlinked, oriented, trivially embedded circles
Welded ribbon tangles

- \( B^4 = B^3 \times [0, 1] \);
- \( k_+ \) circles in the upper copy of \( B^3 \), and \( k_- \) circles in the lower copy of \( B^3 \);
- \( A_1, \ldots, A_k \) embedded annuli and \( E_1, \ldots, E_m \) embedded tori \( E_1, \ldots, E_m \) s.t.:

\[
k = \frac{k_+ + k_-}{2}
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  - \( \partial A_i \) is the disjoint sum of two circles;
  - both annuli and tori admit a filling with 3-balls and solid tori;

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  - singular points are ribbon singularities.

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$k = k_+ + k_-$

The set of ribbon tangles

$rTA_n$: set of ribbon tangles up to ambient isotopy fixing the boundary circles.
Ribbon singularity

Flatly transverse disk whose preimage are two disk:

- one in the interior of a filling,
- the other with interior included in the interior of a filling, and an essential curve as boundary.
Representing welded ribbon tangles
Broken surfaces

Projecting a ribbon tangle’s singularity in $B^3 = B^2 \times I$:

**Warning**

We lose the information about whether the “flying disk” was moving upward or downward!
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**Convention**

Erase a neighbourhood of the tube corresponding to the lower preimage disk.
From ribbon tangles to broken surfaces

Representing ribbon tangles

Any ribbon tangle can be represented by a broken surface diagram.

1. Ribbon tangles and broken surfaces

2. The Alexander functor

3. Welded diagrams and the Tube map

4. A combinatorial approach to the Alexander functor

5. Calculating the Alexander functor with the Alexander polynomial
The category Rib

Objects: sequences of signs $(\varepsilon_1, \ldots, \varepsilon_k)$;
Morphisms: $(\varepsilon^-, \varepsilon^+)$-ribbon tangle with stacking as composition.
Induced coverings

Construction of the functor

Objects of Rib $\rightsquigarrow$ \( R \)-modules (\( R = \mathbb{Z}[t, t^{-1}] \))

Morphisms of Rib $\rightsquigarrow$ Linear maps of degree \( \delta k \) between exterior algebras of \( R \)-modules.

Alexander functor \( \rho \):

\[
(\varepsilon^+_1, \ldots, \varepsilon^+_k) \rightsquigarrow \Lambda(R \text{-- mod})
\]

\((k_-, k_+)\) -- ribbon tangle

\[
(\varepsilon^-_1, \ldots, \varepsilon^-_k) \rightsquigarrow \Lambda(R \text{-- mod})
\]

\[
\Lambda(R \text{-- linear maps of deg } \delta k)
\]
Induced coverings

$$B^3$$

$$\epsilon_i^\pm \in \{-1, +1\}$$

$$(\epsilon_1^+, \epsilon_2^+, \ldots, \epsilon_k^+)$$

$$(\epsilon_1^-, \ldots, \epsilon_k^-)$$
Induced coverings

\[(\epsilon_1^+, \epsilon_2^+, \ldots, \epsilon_{k_+}^+)\]

\[\chi_+: \pi_1(B^3 \setminus \{C_1, \ldots, C_n\}, \ast) \to \mathbb{Z} = \langle t \rangle\]

\[x_i \mapsto t^{\epsilon_i^+}\]

One variable or many variables?

It is possible to take \(\mathbb{Z}^{k_+} = \langle t_1, \ldots, t_{k_+} \rangle\) in order to obtain a multivariable invariant (one variable for each component).
Induced coverings

\[(\varepsilon_1^+, \varepsilon_2^+, \ldots, \varepsilon_{k_+}^+)\]

\[\chi_+: \pi_1(B^3 \setminus \{C_1, \ldots, C_n\}, \ast) \rightarrow \mathbb{Z} = \langle t \rangle\]

\[x_i \mapsto t^{\varepsilon_i^+}\]

This epimorphisms defines a covering \((B^3 \setminus \{C_1, \ldots, C_n\})_{\chi}^+\).

The module \(H_+\)

We define \(H_+\) to be \(H_1((B^3 \setminus \{C_1, \ldots, C_n\})_{\chi}^+, \ast; \mathbb{Z}[t, t^{-1}])\).
The $R$-modules $H_-$ and $H_+$

\[
(\varepsilon_1^+, \varepsilon_2^+, \ldots, \varepsilon_{k_+}^+ ) \rightarrow \Lambda H_+ \\
(\varepsilon_1^-, \ldots, \varepsilon_{k_-}^- ) \rightarrow \Lambda H_-
\]

$B^3$ linear application of degree $\delta k = \frac{k_+ - k_-}{2}$
Let $T$ be a ribbon tangle with $n$ components, we denote:

- the exterior $X_T = B^4 \setminus \text{Tub } (T)$;
- $m_{\pm}$ the inclusion maps of the upper and lower copies of $B^3$ in $X_T$;
- the homology group $H_1(X_T) \simeq \mathbb{Z}^n$ is generated by the meridians of the annuli and tori;
- $\chi$ be the extension of the epimorphisms $\chi_+$ and $\chi_-$;
- $\hat{X}_T$ the maximal abelian cover defined.
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**The module $H$**

We define $H$ to be $H_1(\hat{X}_T, *; \mathbb{Z}[t, t^{-1}])$, where $*$ is a basepoint on $\partial_* B^4$. 
The Alexander function

$R = \mathbb{Z}[t, t^{-1}]$; $M$ a $R$-module of finite type with a deficiency $k$

presentation:

$\langle \gamma_1, \ldots, \gamma_{p+k} \mid r_1, \ldots, r_p \rangle$.

$\Gamma$ = free $R$-module engendred by $\langle \gamma_1, \ldots, \gamma_{p+k} \rangle$.

$\hat{r} = r_1 \wedge \cdots \wedge r_p$ and $\hat{\gamma} = \gamma_1 \wedge \cdots \wedge \gamma_{p+k}$.

The Alexander function

$\varphi_{(M,k)} : \wedge^k M \to R$ is the $R$-linear application defined by:

$u \wedge \hat{r} = \varphi(u) \cdot \hat{\gamma}$

for each $u = u_1 \wedge \cdots \wedge u_k \in \wedge^k M$.

For $k$ fixes, different deficiency $k$ presentations give the Alexander functions

which differ by a multiplicative unitary element of $R$. 
The Alexander functor $\rho$

$\varphi_{(H,k)} : \wedge^k H \to R$ Alexander function, $i_\pm : H_\pm \to H$, $k = \frac{k_+ + k_-}{2}$, $\delta k = \frac{k_+ - k_-}{2}$.

**Alexander invariant**

$\rho_\tau : \wedge (\rho_{i,\tau}) : \wedge H_- \to \wedge H_+$ is defined as follows: for $u_- \in \wedge^i H_-$, $\rho_{i,\tau}(u_-)$ is the element of $\wedge^{i+\delta k} H_+$ that, for each $w_+ \in \wedge^{k-i} H_+$, satisfies:

$$\varphi(H,k)(i_-(u_-) \wedge i_+(w_+)) = \det_+(\rho_{i,\tau}(u_-) \wedge w_+).$$

where $\det_+ : \wedge^{k_+} H_+ \to R$ is a volume form on $H_+$.

$\rightarrow$ Classical case: Bigelow, Cattabriga et Florens (2012).

**Functoriality**

$\rho$ is a functor from the category of 3-dim cobordisms with a representation of their fundamental group to the category of $\mathbb{Z}$-graded $R$-modules.

$\rightarrow$ Classical case: Florens et Massuyeau (2014).
Contents

1. Ribbon tangles and broken surfaces

2. The Alexander functor

3. Welded diagrams and the Tube map

4. A combinatorial approach to the Alexander functor

5. Calculating the Alexander functor with the Alexander polynomial
Welded diagrams

Welded $k$-tangle diagram $T$

Immersion of $k$ oriented arcs and a certain number of circles in $B^2$ such that:

- $\partial I \subset \partial B_2$,
- double points: finite number, transverse, decorated as positive, negative or virtual, modulo generalized Reidemeister moves.
The tube application

For every diagram, one can associate a broken surface, and hence a ribbon tangle by “blowing up” strings as follows:

The tube map

This assignment defines a map $\text{Tube} : \text{diagrams} \to rTA_n$. 
The tube application

For every diagram, one can associate a broken surface, and hence a ribbon tangle by “blowing up” strings as follows:

The tube map

This assignment defines a map $\text{Tube} : \text{diagrams} \rightarrow r\text{TA}_n$.

Proposition (Yanagawa, Satoh - Audoux, Bellingeri, Meilhan, Wagner)

The map $\text{Tube}$ is surjective.
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The Alexander “polynomial” for welded tangle diagrams

Welded tangle diagrams

Circuit algebra structure
The Alexander “polynomial” for welded tangle diagrams

Welded tangle diagrams $\rightarrow$ Circuit algebra structure $\rightarrow$ Another circuit algebra?
A pair of modules defined on welded tangle diagrams

The modules $H_{in}$ and $H_{out}$

$X^{in} = \{a, b, c\}$ \quad $\rightarrow$ \quad $H_{in} = \mathbb{Z}[t^{\pm 1}]$-module over $X^{in}$

$X^{out} = \{d, e, f\}$ \quad $\rightarrow$ \quad $H_{out} = \mathbb{Z}[t^{\pm 1}]$-module over $X^{out}$

Alexander Half Densities

An *Alexander Half Density* of $X^{in}$ and $X^{out}$, is an element of

$$\mathcal{D}(X^{in}, X^{out}) = \wedge^n (H_{in} \oplus H_{out}).$$
A pair of modules defined on welded tangle diagrams
The modules $H_{in}$ and $H_{out}$

\[ X^{in} = \{a, b, c\} \quad \rightarrow \quad H_{in} = \mathbb{Z}[t^{\pm 1}]-\text{module over } X^{in} \]
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**Alexander Half Densities**

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**The circuit algebra of Alexander Half Densities**

Alexander Half Densities with composition (multilinear applications among AHD) form a circuit algebra.
The Alexander matrix

A welded tangle diagram, we can associate a matrix of the form:

\[ A(T) = \begin{pmatrix} \text{Internal arcs} & \mathcal{X}^{\text{in}} & \mathcal{X}^{\text{out}} \\ \mathcal{X}^{\text{out}} & \end{pmatrix} \]
The Alexander polynomial of a welded tangle diagram

We define:

$$\mathcal{A}(T) = \sum_{i_1 < \cdots < i_k} (A(T)^{i_1, \ldots, i_k}) x_{i_1} \wedge \cdots \wedge x_{i_k} \in \wedge^k(H_{in} \oplus H_{out})$$

where $A(T)^{i_1 < \cdots < i_k}$ is the minor of $A$, with respect to the columns corresponding to internal arcs and arcs $i_1, \ldots, i_k$. 
An invariant morphism of circuit algebras

- $\mathcal{A}$ is a morphism between the circuit algebras $\mathcal{T}$ of welded tangle diagrams and $\mathcal{D}$ of Alexander Half Densities;
- $\mathcal{A}$ is an invariant for welded tangle diagrams defined modulo a unit of $\mathbb{Z}[t, t^{-1}]$.

→ Virtual case: Jana Archibald (2010).
Ribbon tangles and broken surfaces

The Alexander functor

Welded diagrams and the Tube map

A combinatorial approach to the Alexander functor

Calculating the Alexander functor with the Alexander polynomial
Proposition

The matrix $A(T)$ is a presentation matrix for $H$ with deficiency $k$. 

<table>
<thead>
<tr>
<th>Internal arcs</th>
<th>$X^{in}$</th>
<th>$X^{out}$</th>
</tr>
</thead>
<tbody>
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<td><strong>Internal arcs</strong></td>
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<td>$X^{out}$</td>
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</tbody>
</table>
For a braid $T$: $k = k_- = k_+$, and $\delta k = \frac{k_+ - k_-}{2} = 0$.

- **Bigelow, Cattabriga, Florens**: $\rho_i : \wedge^i H_- \to \wedge^i H_+$ is the $i$-th external power of the Burau representation, modulo a multiplicative unit of $R$, that depends on the chosen presentation for $H$.

- Chosing $A(T)$ as presentation matrix, we get

$$\rho = \bigoplus_i \rho_i = - \bigoplus_i \wedge \rho_{\text{Burau}}.$$
Theorem (2015 - D., Florens)

Let \( \tau \) be an \((\varepsilon^-, \varepsilon^+)\)-ribbon tangles avec \( k_- \) et \( k_+ \) circles, \( T(\tau) \) welded tangle diagram obtained by projection. There’s a fonctorial isomorphism

\[
\alpha : \wedge^k(H_{in} \oplus H_{out}) \to \text{Hom}_{\delta_k}(\wedge H_-, \wedge H_+)
\]

that sends \( \mathcal{A}(T(\tau)) \) to \( \rho(\tau) \).
**Decomposition**

Partition of the points of a welded tangle diagrams that the Tube map will send to circles “at the top” and “at the bottom”.

**Theorem (2015 - D., Florens)**

Let \( T \) be a welded tangle diagram, and \( \mu \) a decomposition. There’s an isomorphism

\[
\beta : \text{Hom}_{\delta_k}(\wedge H_-, \wedge H_+) \rightarrow \wedge^k (H_- \oplus H_+) \cong \wedge^k (H_{in} \oplus H_{out})
\]

that send \( \rho(\tau_\mu(T)) \) to \( A(T) \). In particular, \( \beta(\rho(\tau_\mu(T))) \) does not depend on the choice of \( \mu \).
Thank you for your attention.