Master Project at EPFL

Tensors, preorders and ultraspaces: constructions stemming from the Vietoris spaces

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20 July 2012

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1 Introduction

If one considers a commutative ring $R$, a right $R$-module $M_1$ and a left $R$-module $M_2$ over $R$ serves in particular to classify the maps from $M_1 \times M_2$ to another $R$-module, that are linear in each variable. This situation can happen in other contexts as modules. In [Sea12], a general context is defined, in which one can express the fact that a morphism has a particular property in each variable. This means that some authors for example [Shm74], [BN76], [KW10] and [Sea12] have given another context than modules in which we also can consider bimorphisms. The context of [Shm74], [KW10] and [BN76] is that of complete lattices. If $L_1$, $L_2$ are complete lattices, the idea is to define the tensor product of $L_1$ and $L_2$ as a complete lattice $L_1 \boxtimes L_2$ with a map $q : L_1 \times L_2 \rightarrow L_1 \boxtimes L_2$ which preserves arbitrary suprema in each variable and such that, for every complete lattice $L_3$ and every map $f : L_1 \times L_2 \rightarrow L_3$ preserving arbitrary suprema in each variable, there exists a unique map $\tilde{f} : L_1 \boxtimes L_2 \rightarrow L_3$ preserving arbitrary suprema and satisfying $\tilde{f} \circ q = f$. In this document, we follow the approach of [Sea12] to present a general context in which we can define tensor products.

To introduce this context we need a monoidal category. The definition of monoidal category can be found in [Bor94, Chapter 6]. In Section 2 Paragraph 2.1, we verify that a category that has all finite products admits a monoidal structure which is given by the product. In Paragraph 2.2, we define monoidal monads, bimorphisms and tensor products.

In Section 3, we present different categories admitting a tensor product. These are, the category of complete lattices Sup, the category of almost complete sup-semilattices Sup0, the category of continuous lattices Cnt and the category of compact Hausdorff spaces CompHaus. In the case of complete lattices, the tensor product is explicitly described.
Given a topological space \((X, \tau)\), some authors, for example [Mic51], [SR70], [Wyl85], have studied a topology on the set of non-empty compact subset of \(X\). In fact it was Felix Hausdorff who defined a metric on the the family of non-empty compact subsets of a metric space. Later, different authors have generalized this idea for a topological space. There are exercises in [Bou71] and in [Eng89] which study this new topological space and some properties inherited. In [Mer10], some of these exercises are solved. Oswald Wyler and Leopold Vietoris studied the case of a compact Hausdorff topological space. We will call the new topological space with the non-empty compact subsets, the Vietoris space [Vie22]. In [Wyl85], a monad \(\mathbb{K} = (K, \{\}, \cup)\) is defined on the category of compact and Hausdorff spaces. The objects \(KK\) are the Vietoris space of \(X\). In fact we can also define this monad on the category of topological spaces.

In Paragraph 3.1, we use the monad \(\mathbb{K}\) and an adjunction between the categories of topological spaces and of preordered spaces. The \(S\)-algebras are described.

In Section 4 we apply several times the functor \(K\) on a topological space. We use an ordinal \(\alpha\) to express the fact of applying '\(\alpha\)'times the functor \(K\). We study some properties that are preserved by limit ordinals and some that are not.

The list of categories and monads considered is at the end of the document.

## 2 Categorical concepts

### 2.1 The product in a category defines a monoidal structure

**Notations**

Let \(\mathcal{C}\) be a category that has every finite product. For \(A, B \in \text{Ob}(\mathcal{C})\), we write \((A \times B, \pi_A, \pi_B)\) for the product of \(A\) and \(B\) and its projections. If \(f : X \to A\) and \(g : X \to B\) are two morphisms, we write \(<f, g> : X \to A \times B\) for the unique morphism such that \(\pi_A \cdot <f, g> = f\) and \(\pi_B \cdot <f, g> = g\).

If we have \(f : A \to A'\) and \(g : B \to B'\), we write \(f \times g := <f \cdot \pi_A, g \cdot \pi_B> : A \times B \to A' \times B'\), which is the unique morphisms satisfying \(\pi_{A'} \cdot (f \times g) = f \cdot \pi_A\) and \(\pi_{B'} \cdot (f \times g) = g \cdot \pi_B\).

**Theorem 2.1**

Let \(\mathcal{C}\) be a category such that every finite product exists, then the product \(\mathcal{C} \times \mathcal{C} \to \mathcal{C}\) is a tensor of a monoidal structure on \(\mathcal{C}\).

**Proof.** Functoriality: Let \(f : A \to A', g : A' \to A''\), \(h : B \to B'\) and \(i : B' \to B''\) be some morphisms and let us prove that \((g \times i) \cdot (f \times h) = g \cdot f \times i \cdot h\). We have \(\pi_{A''} \cdot (g \times i) \cdot (f \times h) = g \cdot \pi_{A'} \cdot (f \times h) = g \cdot f \cdot \pi_A\) and \(\pi_{B''} \cdot (g \times i) \cdot (f \times h) = i \cdot \pi_{B'} \cdot (f \times h) = i \cdot h \cdot \pi_B\). So, by unicity, \((g \times i) \cdot (f \times h) = g \cdot f \times i \cdot h\).

For \(X, Y, Z \in \text{Ob}(\mathcal{C})\), let us consider the morphism \(\sigma_{X,Y,Z} : (X \times Y) \times Z \to X \times (Y \times Z)\) as follows:

$$\begin{align*}
(X \times Y) \times Z &\rightarrow (X \times Y) \times Z \\
X \times Y &\rightarrow (X \times Y) \times Z \\
X &\rightarrow X \\
Y \times Z &\rightarrow (X \times Y) \times Z \\
X \times (Y \times Z) &\rightarrow (X \times Y) \times Z
\end{align*}$$

The morphism \(\sigma_{X,Y,Z}\) is the unique morphism \((X \times Y) \times Z \to X \times (Y \times Z)\) satisfying

$$\begin{align*}
\pi_X \cdot \sigma_{X,Y,Z} &= \pi_X \cdot \pi_{X \times Y}, \\
\pi_Y \cdot \sigma_{X,Y,Z} &= \pi_Y \cdot \pi_{X \times Y}, \\
\pi_Z \cdot \sigma_{X,Y,Z} &= \pi_Z.
\end{align*}$$

Now, let us prove that \(\sigma\) is natural in \(X, Y\) and \(Z\), i.e. if \(f : X \to X', g : Y \to Y'\) and \(h : Z \to Z'\) are morphisms then
the following diagram commutes:

\[
\begin{array}{ccc}
(X \times Y) \times Z & \xrightarrow{(f \times g) \times h} & (X' \times Y') \times Z' \\
\sigma_{X,Y,Z} & & \sigma_{X',Y',Z'} \\
X \times (Y \times Z) & \xrightarrow{f \times (g \times h)} & X' \times (Y' \times Z')
\end{array}
\]

By universality of the product, to show that \((f \times (g \times h)) \cdot \sigma_{X,Y,Z} = \sigma_{X',Y',Z'} \cdot ((\tilde{f} \times \tilde{g}) \times h)\), it is sufficient to show that \(\pi_X \cdot (f \times (g \times h)) \cdot \sigma_{X,Y,Z} = \pi_{X'} \cdot \sigma_{X',Y',Z'} \cdot ((\tilde{f} \times \tilde{g}) \times h)\) and \(\pi_{Y' \times Z} \cdot (f \times (g \times h)) \cdot \sigma_{X,Y,Z} = \pi_{Y' \times Z} \cdot \sigma_{X',Y',Z'} \cdot ((\tilde{f} \times \tilde{g}) \times h)\); Using (1), (2), (3), we have

\[
\begin{align*}
\pi_X \cdot (f \times (g \times h)) \cdot \sigma_{X,Y,Z} &= f \cdot \pi_X \cdot \sigma_{X,Y,Z} = f \cdot \pi_X \cdot \sigma_{X,Y,Z}, \\
\pi_{X'} \cdot \sigma_{X',Y',Z'} \cdot ((\widetilde{f} \times \widetilde{g}) \times h) &= \pi_{X'} \cdot \sigma_{X',Y',Z'} \cdot ((\widetilde{f} \times \widetilde{g}) \times h) = \pi_{X'} \cdot (f \times (g \times h)) \cdot \sigma_{X,Y,Z} = \pi_{X'} \cdot (f \times (g \times h)) \cdot \sigma_{X,Y,Z}, \\
\pi_{Y'} \cdot \sigma_{Y',X',Z'} \cdot \sigma_{X',Y',Z'} &= \pi_{Y'} \cdot (g \times h) \cdot \pi_{Y',X',Z'} = \pi_{Y'} \cdot (g \times h) \cdot \pi_{Y',X',Z'}, \\
\pi_{Y' \times Z} \cdot \sigma_{Y',X',Z'} \cdot \sigma_{X',Y',Z'} &= \pi_{Y' \times Z} \cdot \sigma_{Y',X',Z'} \cdot \sigma_{X',Y',Z'} = \pi_{Y' \times Z} \cdot \sigma_{Y',X',Z'} \cdot \sigma_{X',Y',Z'}, \\
\pi_{Z'} \cdot \sigma_{Z',Y',X'} \cdot \sigma_{Y',X',Z'} &= \pi_{Z'} \cdot \sigma_{Z',Y',X'} \cdot \sigma_{Y',X',Z'} = \pi_{Z'} \cdot \sigma_{Z',Y',X'} \cdot \sigma_{Y',X',Z'} = \pi_{Z'} \cdot \sigma_{Z',Y',X'} \cdot \sigma_{Y',X',Z'} = \pi_{Z'} \cdot \sigma_{Z',Y',X'} \cdot \sigma_{Y',X',Z'} = \pi_{Z'}.
\end{align*}
\]

So \((f \times (g \times h)) \cdot \sigma_{X,Y,Z} = \sigma_{X',Y',Z'} \cdot ((\tilde{f} \times \tilde{g}) \times h)\).

Now, let us prove that \(\sigma_{X,Y,Z}\) is an isomorphism with inverse \(\tau_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z\) defined as follows:

\[
\begin{array}{ccc}
X \times (Y \times Z) & \xrightarrow{\nabla_{X,Y,Z} = 1_X \times \pi_Y} & X \times Y \\
\pi_X & & \pi_Y \\
X \times Y & \xrightarrow{\pi_Z} & Z \\
X \times Y & \xrightarrow{\tau_{X,Y,Z} = (X \times Y) \times Z} & (X \times Y) \times Z.
\end{array}
\]

The morphism \(\tau_{X,Y,Z}\) is the unique morphism: \(X \times (Y \times Z) \to (X \times Y) \times Z\) satisfying

\[
\begin{align*}
\pi_X \cdot \pi_X \cdot \tau_{X,Y,Z} &= \pi_X, \\
\pi_Y \cdot \pi_X \cdot \tau_{X,Y,Z} &= \pi_Y \cdot \pi_Y \times Z, \\
\pi_Z \cdot \tau_{X,Y,Z} &= \pi_Z \cdot \pi_Y \times Z.
\end{align*}
\]

By universality, to prove that \(\tau_{X,Y,Z} \cdot \sigma_{X,Y,Z} = 1_{(X \times Y) \times Z}\) it is sufficient to prove that \(\pi_X \cdot \pi_X \cdot \tau_{X,Y,Z} \cdot \sigma_{X,Y,Z} = \pi_X \cdot \pi_Y \times Z\) and \(\pi_Z \cdot \tau_{X,Y,Z} \cdot \sigma_{X,Y,Z} = \pi_Z\). Using (1), (2), (3), (4), (5), (6) we have

\[
\begin{align*}
\pi_X \cdot \pi_X \cdot \tau_{X,Y,Z} \cdot \sigma_{X,Y,Z} &= \pi_X \cdot \pi_X \cdot \sigma_{X,Y,Z} = \pi_X \cdot \pi_X \cdot \pi_Y \times Z, \\
\pi_Y \cdot \pi_X \cdot \tau_{X,Y,Z} \cdot \sigma_{X,Y,Z} &= \pi_Y \cdot \pi_Y \times Z \cdot \sigma_{X,Y,Z} = \pi_Y \cdot \pi_Y \times Z, \\
\pi_Z \cdot \tau_{X,Y,Z} \cdot \sigma_{X,Y,Z} &= \pi_Z \cdot \pi_Y \times Z \cdot \sigma_{X,Y,Z} = \pi_Z.
\end{align*}
\]

Similarly, one checks that \(\sigma_{X,Y,Z} \cdot \tau_{X,Y,Z} = 1_{(X \times Y) \times Z}\).

- Since every finite product exists in \(\mathcal{C}\), there is a terminal object \(T\) (the product on the empty family). We claim that \(T\) is the unit object. For each \(X \in \text{Ob}(\mathcal{C})\), we have the morphism \(\lambda_X := \pi_X : T \times X \to X\) and \(\rho_X := \pi_X : X \times T \to X\). Let us call \(1_X\) the unique morphism \(X \to T\). We claim that \(\lambda_X\) and \(\rho_X\) are isomorphisms with inverse \(<1_X,1_X>\) and \(<1_X,1_X>\), respectively. We have

\[
\begin{align*}
\lambda_X : <1_X,1_X> &= \pi_X \cdot <1_X,1_X> = 1_X, \\
\pi_T : <1_X,1_X> \cdot \lambda_X &= 1_{T \times X}, \\
\pi_X : <1_X,1_X> \cdot \lambda_X &= \lambda_X = \pi_X.
\end{align*}
\]
So $1_X \cdot \lambda_X = 1_{T \times X}$. In the same way we prove that $\rho_X$ is an isomorphism.

Now let us prove that $\lambda_X$ is natural in $X$, i.e. if $f : X \to X'$ is a morphism in $\mathcal{C}$ then the following diagram commutes:

$$
\begin{array}{c}
T \times X \xrightarrow{\lambda_X} X \\
\downarrow_{1_T \times f} \\
T \times X' \xrightarrow{\lambda_X'} X'.
\end{array}
$$

This is because $\lambda_X' \cdot (1_T \times f) = \pi_X' \cdot (1_T \times f) = f \cdot \pi_X = f \cdot \lambda_X$. In the same way, we prove that $\rho_X$ is natural in $X$.

- Let us prove that the following diagram commutes for any $X, Y \in \text{Ob}(\mathcal{C})$:

$$
\begin{array}{c}
(X \times T) \times Y \xrightarrow{\sigma_{X,T,Y}} X \times (T \times Y) \\
\downarrow_{\rho_X \times 1_Y} \\
X \times Y.
\end{array}
$$

To do this, one checks that $\pi_X \cdot (1_X \times \lambda_Y) \cdot \sigma_{X,T,Y} = \pi_X \cdot (\rho_X \times 1_Y)$ and $\pi_Y \cdot (1_X \times \lambda_Y) \cdot \sigma_{X,T,Y} = \pi_Y \cdot (\rho_X \times 1_Y)$. Using the formulas (1), (2), (3), we have

$$
\begin{align*}
\pi_X \cdot (1_X \times \lambda_Y) \cdot \sigma_{X,T,Y} &= \pi_X \cdot \sigma_{X,T,Y} = \pi_X \cdot \pi_{X,T}, \\
\pi_X \cdot (\rho_X \times 1_Y) &= \rho_X \cdot \pi_{X,T} = \pi_X \cdot \pi_{X,T}, \\
\pi_Y \cdot (1_X \times \lambda_Y) \cdot \sigma_{X,T,Y} &= \lambda_Y \cdot \pi_{T \times Y} \cdot \sigma_{X,T,Y} = \pi_Y \cdot \pi_{T \times Y} \cdot \sigma_{X,T,Y} = \pi_Y, \\
\pi_Y \cdot (\rho_X \times 1_Y) &= \pi_Y.
\end{align*}
$$

- Let us prove that the following diagram commutes for any $A, B, C, D \in \text{Ob}(\mathcal{C})$:

$$
\begin{array}{c}
(A \times B) \times (C \times D) \\
\downarrow_{\sigma_{A \times B, C \times D}} \\
((A \times B) \times C) \times D \\
\downarrow_{\sigma_{A, B \times C} \times 1_D} \\
(A \times (B \times C)) \times D \xrightarrow{\sigma_{A, B \times C \times D}} A \times ((B \times C) \times D).
\end{array}
$$

To do this one proves that

$$
\pi_A \cdot (A \times B, C \times D) \cdot \sigma_{A \times B, C \times D} = \pi_A \cdot (1_A \times \sigma_{B, C} \times D) \cdot (A \times B, C \times 1_D),
$$

and

$$
\pi_{B \times (C \times D)} \cdot (A \times B, C \times D) \cdot \sigma_{A \times B, C \times D} = \pi_{B \times (C \times D)} \cdot (1_A \times \sigma_{B, C} \times D) \cdot (A \times B, C \times 1_D).
$$

Using formulas (1), (2), (3), we have the following:

$$
\begin{align*}
\pi_A \cdot (A \times B, C \times D) \cdot \sigma_{A \times B, C \times D} &= \pi_A \cdot (A \times B) \cdot (A \times B, C \times D) \\
&= \pi_A \cdot (A \times B) \cdot \pi_{(A \times B) \times C}, \\
\pi_A \cdot (1_A \times \sigma_{B, C} \times D) \cdot (A \times B, C \times 1_D) &= \pi_A \cdot (A \times B, C \times D) \cdot (A \times B, C \times 1_D) \\
&= \pi_A \cdot (A \times B, C \times 1_D) \\
&= \pi_A \cdot (A \times B, C \times 1_D) \\
&= \pi_A \cdot A \times B \cdot \pi_{(A \times B) \times C}.
\end{align*}
$$
2.2 Tensor product of algebras over a monoidal monad

The goal of this section is to give a generalization of the tensor product of bi $R$-modules over a commutative ring $R$. One can see a module over a ring $R$ as an algebra over $	ext{Set}$ for the free $R$-module monad. This is the way that we will follows. If one sees the $R$-modules $M_1$, $M_2$ and $M_3$ as $M_R$-algebras, where $M_R$ is the free $R$-module monad, one says that $f : M_1 \times M_2 \to M_3$ is a bimorphism if the following diagram commutes:

$$
\begin{array}{ccc}
FM_1 \times FM_2 & \xrightarrow{\kappa_{M_1,M_2}} & F(M_1 \times M_2) \\
M_1 \times M_2 & \xrightarrow{f} & M_3 \\
\end{array}
$$

The vertical arrows are the algebra structures. The map $\kappa_{M_1,M_2}$ is defined by $\kappa_{M_1,M_2}(\Sigma_i r_i m_{i,1}, \Sigma_j r_j m_{j,2}) = \Sigma_{i,j}(r_i + r_j)(m_{i,1} + m_{j,2})$.

The tensor product $M_1 \boxtimes M_2$ of $M_1$ and $M_2$ is an $R$-module with a bimorphism $q : A \times B \to A \boxtimes B$ such that for every bimorphism $f : M_1 \times M_2 \to M_3$, there exists a unique morphism $\hat{f}$ of $R$-modules $A \boxtimes B \to M_3$ such that $\hat{f} \circ q = f$. We know that it exists and how to construct it if $M_1$ and $M_2$ are $R$-modules.

Now we will generalize this. We follow the approach from [Sep2]. Let $(\mathcal{C}, \otimes)$ be a monoidal category with unit $E$ and with natural isomorphisms

$$
\sigma_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \quad \lambda_X : E \otimes X \to X, \quad \rho_X : X \otimes E \to X.
$$

Let $T = (T, \eta, \mu)$ be a monad over $\mathcal{C}$. 
Definition
We say that the monad $T$ is **monoidal** over $(\mathcal{C}, \otimes)$ if it comes with a family of maps $\kappa_{X,Y} : TX \otimes TY \to T(X \otimes Y)$ which are natural in $X$ and $Y$ and which are compatible with the monoidal structure $\otimes$ and the monad $T$. This means that all of the following diagrams commute for every objects $X,Y,Z$ in $\mathcal{C}$:

1. 
\[
\begin{array}{c}
(TX \otimes TY) \otimes TZ & \xrightarrow{\kappa_{X,Y} \otimes 1_{TZ}} & T(X \otimes Y) \otimes TZ & \xrightarrow{\kappa_{(X \otimes Y),Z}} & T((X \otimes Y) \otimes Z) \\
\sigma_{TX,TY,TZ} & & T \sigma_{X,Y,Z} & & \\
TX \otimes (TY \otimes TZ) & \xrightarrow{1_{TX} \otimes \kappa_{Y,Z}} & TX \otimes T(Y \otimes Z) & \xrightarrow{\kappa_{X,(Y \otimes Z)}} & T(X \otimes (Y \otimes Z)) ,
\end{array}
\]

2. 
\[
\begin{array}{c}
E \otimes TX & \xrightarrow{\eta_E \otimes 1_{TX}} & TE \otimes TX & \xrightarrow{\kappa_{E,X}} & T(E \otimes X) \\
\lambda_{TX} & & TX & & \\
E \otimes TX & \xrightarrow{1_{TX} \otimes \eta_E} & TX \otimes TE & \xrightarrow{\kappa_{X,E}} & T(X \otimes E) \\
\rho_{TX} & & TX , & & \\
\end{array}
\]

3. 
\[
\begin{array}{c}
TX \otimes TY & \xrightarrow{\kappa_{X,Y}} & T(X \otimes Y) ,
\end{array}
\]

4. 
\[
\begin{array}{c}
TTX \otimes TTY & \xrightarrow{T \sigma_{X,Y,TX,TY}} & TT(X \otimes Y) \\
\mu_{X \otimes Y} & & TX \otimes TY & \xrightarrow{\kappa_{X,Y}} & T(X \otimes Y) .
\end{array}
\]

From now on, we suppose that we have a monoidal category $\mathcal{C}$ with a monoidal monad $T$. We keep the notation $\kappa$.

Definition
Let $(A,\alpha), (B,\beta)$ and $(C,\gamma)$ be $T$-algebras and $f : A \otimes B \to C$ be a $\mathcal{C}$-morphism. We say that $f$ is a **bimorphism** if the following diagram commutes:
\[
\begin{array}{ccc}
TA \otimes TB & \xrightarrow{\kappa_{A,B}} & T(A \otimes B) & \xrightarrow{Tf} & TC \\
\alpha \otimes \beta \downarrow & & \gamma \downarrow & & \\
A \otimes B & \xrightarrow{f} & C .
\end{array}
\]

Definition
Let $(A,\alpha)$ and $(B,\beta)$ be two $T$-algebras. We say that $(A \boxtimes B, \boxtimes)$ is the **tensor product** of $(A,\alpha)$ and $(B,\beta)$ if there exists $q_{A,B} : T(A \boxtimes B) :\to A \boxtimes B$ such that the diagram
\[
\begin{array}{ccc}
T(TA \otimes TB) & \xrightarrow{\mu_{A,B} T(\kappa_{A,B})} & T(A \otimes B) & \xrightarrow{q_{A,B}} & A \boxtimes B \\
\mu_{A,B} T(\kappa_{A,B}(1_A \otimes \eta_B \otimes \beta)) & & & & \\
\mu_{A,B} T(\kappa_{A,B}(1_A \otimes \eta_B \otimes \beta)) & & & & \\
T(TA \otimes TB) & \xrightarrow{\mu_{A,B} T(\kappa_{A,B}(1_A \otimes \eta_B \otimes \beta))} & T(A \otimes B) & \xrightarrow{q_{A,B}} & A \boxtimes B
\end{array}
\]

is a coequalizer in $\mathcal{C}^T$.

Remark 2.2
The tensor product of two arbitrary $T$-algebras does not necessarily exist, but if it exists then it is unique up to $\mathcal{C}^T$-isomorphism.

This is what we want for classifying the bimorphisms as say Lemma 2.3.3 and Proposition 2.3.4 in [Sea12]. From this lemma we have the following.

Proposition 2.3
Let $(A,\alpha)$ and $(B,\beta)$ be two $T$-algebras such that $(A \boxtimes B, \boxtimes)$ exists. For every $T$-algebra $(C,\gamma)$ and every bimorphism $f : A \otimes B \to C$, there exists a unique $\mathcal{C}^T$-morphism $\tilde{f} : A \boxtimes B \to C$ such that $\tilde{f} \cdot q = f$.
\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{f} & C \\
\downarrow q & & \\
A \boxtimes B .
\end{array}
\]
Another idea would have been to define the tensor product by the universal property in Proposition 2.3. The advantage of the definition with the coequalizer is that even we do not necessarily know how to construct it, we know that it is a colimit. In particular, if we know that the category \( C \) is cocomplete, we know that the tensor product exists.

With Proposition 3.4 in [BW05] one knows that a category that is equivalent to the category of algebras on a monad on \( \mathsf{Set} \) is cocomplete.

### 2.3 The monad inherited from an adjunction to a category equipped with a monad

Given two adjunctions \( L \dashv R : \mathcal{D} \to \mathcal{C} \) and \( L' \dashv R' : \mathcal{D} \to \mathcal{C} \), one can compose to have an adjunction \( L' L \dashv R' R : \mathcal{D} \to \mathcal{C} \). If \( \eta : 1_\mathcal{D} \Rightarrow RL \) denotes the unit of the first adjunction, \( \epsilon : LR \Rightarrow 1_\mathcal{D} \) denotes the counit of the first adjunction, \( \eta' : 1_\mathcal{D} \Rightarrow R'L' \) denotes the unit of the second adjunction, \( \epsilon' : L'R' \Rightarrow 1_\mathcal{D} \) denotes the counit of the second adjunction, then the unit of the composition of the two adjunctions is given by \( \eta'' = R\eta' L \cdot \eta : 1_\mathcal{D} \Rightarrow RR'L' \mathcal{L} \mathcal{C} \) and the counit of the composition of the two adjunctions is given by \( \epsilon'' = \epsilon' \cdot L' \epsilon R' : L' L R R' \Rightarrow 1_\mathcal{D} \). This means that if \( C \in \mathcal{C} \), \( \eta''_C = R(\eta'_{L'C}) \cdot \eta_C : C \to RR'L' \mathcal{L} \mathcal{C} \) and if \( E \in \mathcal{D} \), \( \epsilon''_E = \epsilon'_E \cdot L'(R \epsilon_E) : L' L R R' \mathcal{E} \to \mathcal{E} \).

It is well known (see for example [Bor94, chapter 4]) that if \( T = (T, \eta', \mu) \) is a monad on a category \( \mathcal{D} \) then we have the adjunction \( \text{Forget} \dashv \mathbb{T} : \mathcal{D} \to \mathcal{C} \). The unit of this adjunction is \( \eta \) and the counit \( \epsilon \) is given by \( \epsilon(C, \gamma) = \gamma = (TC, \mu) \to (C, \gamma) \). Conversely, an adjunction \( L'' \dashv R'' : \mathcal{D} \to \mathcal{C} \) gives rise to a monad on \( \mathcal{C} \). The functor of the monad is given by \( R'' L'' \). If \( \eta'' \) denotes the unit of this adjunction and \( \epsilon'' \) denotes the counit of this adjunction, then the unit of the monad is \( \eta'' \) and the multiplication of the monad is given by \( R'' \epsilon'' L'' \).

We now consider the case where we have an adjunction \( L \dashv R : \mathcal{D} \to \mathcal{C} \) and a monad on \( \mathcal{D} \) and we will study the monad obtained from the composition of the two adjunctions.

Let us consider the following adjunction, where the unit of the first adjunction is \( \eta : 1_\mathcal{D} \Rightarrow RL \), the counit of the first adjunction is \( \epsilon : LR \Rightarrow 1_\mathcal{D} \) and the monad \( \mathbb{T} \) is \( (T, \eta', \mu) \):

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{L} & \mathcal{D} \\
\downarrow R & & \downarrow \text{Forget} \\
\mathbb{T} & \xleftarrow{\text{Free}} & \mathcal{C}
\end{array}
\]

This gives the adjunction \( \text{Free} L \dashv \text{R Forget} : \mathcal{D} \to \mathcal{C} \) where the unit \( \eta'' = R\eta' \mathcal{L} L \cdot \eta \) and the counit is \( \epsilon'' = \epsilon' \cdot \text{Free \text{Forget}.} \)

Thus the induced monad \( \mathbb{T}' = (T', \eta'', \mu'') \) on \( \mathcal{C} \) is given by

\[
\mathbb{T}' = (R\text{ForgetFree} L, R\eta' \mathcal{L} L \cdot \eta, R\text{Forget}(\epsilon' \cdot \text{Free}(\epsilon')\text{Forget})\text{Free} L).
\]

Explicitly, if \( C \in \mathcal{C} \), then \( R\text{ForgetFree} L C = R\tau C \cdot L \mathcal{C} \) is \( \eta''_C = R(\eta'_{\mathcal{L} C}) \cdot \eta_C \) and \( \epsilon''_C : R\tau L \mathcal{L} R\tau L \mathcal{C} \to R\tau C \) is

\[
\begin{align*}
\epsilon''_C &= R\text{Forget}(\epsilon' \cdot \text{Free}(\epsilon')\text{Forget}) \cdot \mathcal{L} (\mu_L, \mathcal{L} \mu) \\
&= R\text{Forget}(\mu_L \cdot \mathcal{L} \mu) \\
&= R\text{Forget}(\mathcal{L} \mathcal{L} \mu \cdot \mathcal{L} \mathcal{L} \mathcal{L} \mu).
\end{align*}
\]

### 3 Preordered sets

By a preorder relation on a set, we mean reflexive and transitive relation. By an order relation on a set, we mean a reflexive, transitive and antisymmetric relation. If \( X \) is a preordered set (or a poset), we usually write \( \leq \) the preorder (or order) relation.

The category of preordered sets with increasing maps is written \( \mathsf{Preord} \). The category of posets with increasing maps is written \( \mathsf{Poset} \).

In the case of a poset \( X \), if the supremum of \( X' \cup X \) exists, we write it \( \bigvee X' \) and if the infimum of \( X' \) exists, we write it \( \bigwedge X' \). If \( \bigvee \emptyset \) exists, we write it \( 0 \).

In a poset \( (X, \leq) \), an up-directed set \( X' \subset X \) is a non-empty subset such that for all \( x, y \in X' \), there exists \( z \in X' \) satisfying \( x \leq z \) and \( y \leq z \). It follows by induction that if \( X' \) is an up-directed set, then every non-empty finite family of elements in \( X' \) admits an upper bound in \( X' \).

We define a functor \( \mathsf{Topol} : \mathsf{Preord} \to \mathsf{Top} \) as follows. For a preordered set \( (X, \leq) \),

\[
\mathsf{Topol}(X, \leq) = (X, \tau_\leq),
\]

where \( \tau_\leq = \{ U \subset X \mid (x \in U \text{ and } x \leq y \implies y \in U) \} \). One checks easily that it is a topology. This topology is called the Alexandrov topology of \( (X, \leq) \). For an increasing map \( f : (X, \leq) \to (X', \leq') \) between two preordered sets, we define \( \mathsf{Topol}(f) = f \). Let us prove that \( f \) is continuous \( (X, \tau_\leq) \to (X', \tau_{\leq'}) \). Let \( U \in \tau_\leq \), and let us prove that \( f^{-1}(U) \in \tau_{\leq'} \). Let \( x \in f^{-1}(U) \) and let \( y \geq x \). Since \( f \) is increasing, \( f(x) \leq f(y) \). Since \( f(x) \in U \) and \( U \in \tau_{\leq'} \), \( f(y) \in U \). So \( y \in f^{-1}(U) \). This
proves that $\text{Topol}$ is a functor $\text{Preord} \to \text{Top}$.

Now we define a functor $\text{Preorder}: \text{Top} \rightleftarrows \text{Preord}$ as follows. For a topological space $(Y, \tau)$ we define

$$\text{Preorder}(Y, \tau) = (Y, \leq),$$

where $\leq$ is defined by $x \leq y$ iff for all $U \in \tau$, $x \in U$ implies $y \in U$. The reflexivity and the transitivity of the relation $\leq$ are clear. For a continuous map $g: (Y, \tau) \to (Y', \tau')$ between two topological spaces, we define $\text{Preorder}(g) = g$. Let us prove that $g$ is increasing $(Y, \leq) \to (Y', \leq')$. Let $x, y \in Y$ such that $x \leq y$ and let us prove that $f(x) \leq' f(y)$. Let $U \in \tau'$ such that $f(x) \in U$. Since $f$ is continuous, $f^{-1}(U) \in \tau$. So $x \in f^{-1}(U)$ implies $y \in f^{-1}(U)$ and so $f(y) \in U$. This proves that $\text{Preorder}$ is a functor $\text{Preord} \to \text{Top}$.

These two functors are defined in [Sco72].

**Remark 3.1**

If $(X, \tau)$ is a topological space, to check that $x \leq y$, it is sufficient to check the definition of $\leq$ with a sub-basis of $\tau$.

**Lemma 3.2**

The functor $\text{Topol}: \text{Preord} \to \text{Top}$ is left adjoint to the functor $\text{Preorder}: \text{Top} \to \text{Preord}$.

**Proof.** Let $(X, \leq)$ be a preordered set, $(Y, \tau)$ be a topological space and let us prove that we have the following bijection.

$$\text{Top}(\text{Topol}(X, \leq), (Y, \tau)) \cong \text{Preord}((X, \leq), \text{Preorder}(Y, \tau)).$$

Assume $f \in \text{Top}(\text{Topol}(X, \leq), (Y, \tau))$ and let us prove that $f \in \text{Preord}((X, \leq), \text{Preorder}(Y, \tau))$. Let $x, y \in X$ such that $x \leq y$. Let $U \in \tau$ such that $f(x) \in U$. Since $f$ is continuous, $f^{-1}(U) \in \tau$_. So, we have $y \in f^{-1}(U)$. And so $f(y) \in U$. This proves that $f(x) \leq f(y)$.

Assume $f \in \text{Preord}((X, \leq), \text{Preorder}(Y, \tau))$ and let us prove that $f \in \text{Top}(\text{Topol}(X, \leq), (Y, \tau))$. Let $U \in \tau$, $x \in f^{-1}(U)$ and $y \geq x$. Since $f$ is increasing, $f(x) \geq y$. So since $f(x) \in U$, we have $f(y) \in U$. So $y \in f^{-1}(U)$ and so $f^{-1}(U) \in \tau$_. This bijection is clearly natural in $(X, \leq)$ and in $(Y, \tau)$ because the bijection and the functors do not change the morphisms seen as set maps.

**Remark 3.3**

If $(X, \leq)$ is a preordered set then $(X, \leq) = (X, \leq_{\leq})$. Indeed, if $x \leq y$ and $x \in U$ for $U \in \tau$ then $y \in U$. So $x \leq_{\leq} y$.

Conversely, if $x \leq_{\leq} y$, then the open set $\{z \in X \mid x \leq z\}$ contains $y$. So $x \leq y$. This proves that the unit of the adjunction in Lemma 3.8 is an isomorphism.

**Remarks 3.4**

If $(Y, \tau)$ is a topological space then $\tau \subset \tau_{\leq}$. Indeed, let $U \in \tau$, $x \in U$ and $y \geq x$. Since $x \in U$, we have $y \in U$. So $U \in \tau_{\leq}$.

In general for a topological $(Y, \tau)$, $\tau_{\leq} \not\subset \tau$. For example, consider a $T_1$ topological space that is not discrete (so necessary infinite), for example $\mathbb{R}$ with the usual topology. In this case, the topology $\tau_{\leq}$ will be discrete. But if $(X, \leq)$ is a preordered set, by remark 3.3, $\tau_{\leq} = \tau_{\leq}$. In particular, if $A \subset X$ is compact for $\tau_{\leq}$, then it is also compact for $\tau_{\leq}$.

**Remark 3.5**

If $(X, \leq)$ is a preordered set then the family of sets $\uparrow x = \{y \in X \mid x \leq y\}$ for $x \in X$ is a basis for the topology $\tau_{\leq}$. So a subset $X' \subset X$ is compact for $\tau_{\leq}$ if and only if for every $I \subset X$ such that $X' \subset \bigcup_{x \in I} \uparrow x$, there exists $I' \subset I$ finite such that $X' \subset \bigcup_{x \in I'} \uparrow x$.

**Remark 3.6**

The functor $\text{Topol}$ restricts and co-restricts to a functor $\text{Poset} \to T_0\text{Top}$. Indeed, to see that $\text{Topol}(X, \leq)$ is a $T_0$ topological space if $\leq$ is an order relation, let us remark that if $x \not\leq y$, then the set $\{z \in X \mid x \leq z\}$ is open, contains $x$ and does not contains $y$. We keep the notation $\text{Topol}$ for this functor.

**Remarks 3.7**

The functor $\text{Preorder}$ restricts and co-restricts to a functor $\text{Ord}: T_0\text{Top} \to \text{Poset}$. The fact that $\leq$ is antisymmetric comes form the fact that the topology $\tau$ is $T_0$.

In the same way as in the proof of Lemma 3.2 we have the following.

**Lemma 3.8**

The functor $\text{Topol}: \text{Poset} \to T_0\text{Top}$ is left adjoint to the functor $\text{Ord}: T_0\text{Top} \to \text{Poset}$.
3.1 A new monad on the category of preordered sets

We now want to define a monad on $\text{Preord}$ using Lemma 3.2 and paragraph 2.3. We use the $K$ monad on $\text{Top}$ described in Appendices A.1. Using this monad, Lemma 3.2 and paragraph 2.3, we describe the monad $S = (S, \delta, \nu)$ on $\text{Preord}$ induced by $K$ and the adjunction in Lemma 3.2. For a preordered set $(X, \leq)$, we have $S(X, \leq) = \text{Preorder}_{\text{Top}}(X, \leq) = (\mathcal{K}X, \leq_{\mathcal{K}})$. The unit $\delta$ is given by the map $\delta_{(X, \leq)} : X \to \mathcal{K}X, x \mapsto \{x\}$. The multiplication $\nu$ is given by the map $\nu_{(X, \leq)} : \mathcal{K}X \to \mathcal{K}X, M \mapsto \bigcup M$.

In the following examples, we adopt the notation of Hasse diagrams: for representing a preordered set, we write an element below another if it is $\leq$ with a line between them. If two elements are each $\leq$ than the other, we write them as the same level with a line between them.

**Example 3.9**
Let us consider the preordered set $X = \{0, 1\}$ with $0 \neq 1$ and $0 \leq 1$. We have $\mathcal{K}X = \{\{0\}, \{1\}, \{0, 1\}\}$. Then in $S X$ we have

$$\{0\} \leq_{\mathcal{K}} \{0, 1\} \leq_{\mathcal{K}} \{1\}.$$ 

Indeed the family of open sets $\tau_{\leq} = \emptyset, X, \{1\}$. So the family of open sets $\tau_{\leq}$ is

$$\tau_{\leq} = \emptyset, \mathcal{K}X, W(\{1\}), W(X, \{1\}) = \emptyset, \mathcal{K}X, \{\{1\}\}, \{\{0, 1\}, \{1\}\}.$$ 

Then use the definition of the functor $\text{Preorder}$ to deduce the preorder.

**Example 3.10**
Let us consider the preordered set $X = \{0, a, 1\}$ where $0, a, 1$ are distinct and $0 \leq a \leq 1$. Then in $S X$, one has

**Examples 3.11**
Let us consider the following preordered set $X$:

Then in $S X$, one has
Similarly, if $Y$ is the following preordered set,

\[ \begin{array}{c}
  c \\
  a & b \\
\end{array} \]

in $SY$ one has

\[ \begin{array}{c}
  \{c\} \\
  \{a,c\} & \{b,c\} \\
  \{a,b,c\} \\
  \{a\} & \{a,b\} & \{b\} \\
\end{array} \]

**Example 3.12**

Let us consider the following preordered set $X$:

\[ \begin{array}{c}
  1 \\
  x & y \\
  0 \\
\end{array} \]

Then in $SX$, one has

\[ \begin{array}{c}
  \{1\} \\
  \{x,1\} & \{y,1\} \\
  \{1,x,y\} \\
  \{x\} & \{x,y\} & \{1\} \\
  \{0,1\} & \{0,x,1\} & \{0,y,1\} & \{0,x,y,1\} \\
  \{0\} \\
\end{array} \]

**Remark 3.13**

Examples 3.10 and 3.12 show that in general, the functor $K$ does not preserve $T_0$ topological spaces and the functor $S$ does not preserve the antisymmetry of the preorder. So we can not directly use Lemma 3.8 to construct a monad on $\text{Poset}$ in the same way as the monad $S$. 
3.9 and 3.10 allow us to describe all the algebra structures (for $\mathcal{S}$) on $X = \{0, 1\}$, $0 < 1$. There are two algebras which are

\[
\begin{align*}
\{1\} & \rightarrow \{1\} \\
\{0, 1\} & \rightarrow \{0, 1\} \\
\{0\} & \rightarrow \{0\}.
\end{align*}
\]

In [Mer10] Lemme 4.19 shows that if $(X, \tau)$ is a topological space and $x \in X$ and $A \subset X$ is compact for $\tau$, then the set $E = \{\{x, a\} | a \in A\}$ is compact for $\tau^1$. With remark 3.4 we have the following.

**Lemma 3.15**

Let $(X, \leq)$ be a preordered set. Let $x \in X$ and $A \subset X$ be compact for $\tau_{\leq}$. Then the set

\[
\{\{x, a\} | a \in A\}
\]

is compact for $\tau_{\leq}^1$.

With this Lemma one has the following.

**Lemma 3.16**

Let $((X, \leq), \alpha)$ be a preordered set such that $\alpha : S(X, \tau) \rightarrow (X, \tau)$ defines an algebra structure for the monad $\mathcal{S}$. Then $\alpha$ defines an order $\sqsubseteq$ on $X$ by $x \sqsubseteq y \iff \alpha(x, y) = y$. Moreover for every non-empty compact subset $A$ of $X$ (in the sense of $\tau_{\leq}$), the supremum for $\sqsubseteq$ exists and is given by $\alpha(A)$. Finally, if $u \leq v$ then $u \leq \alpha([u, v]) \leq v$.

**Proof.** Let us prove that the relation $\sqsubseteq$ defines an order on $X$. Since $(X, \alpha)$ is an algebra on $\mathcal{S}$, we know that $\alpha \circ \delta_X = 1_X$ so, for $x \in X$ we have $\alpha \circ \delta_X(x) = \alpha(\{x\}) = x$. With the axioms of an $\mathcal{S}$-algebra, we have

\[
\alpha(\{x\}) = x, \quad \alpha \circ \alpha = \alpha \circ \nu_X.
\]

Let $x, y, z \in X$.

- **Reflexivity:** $\alpha(\{x, x\}) = \alpha(\{x\}) = x$ so $x \sqsubseteq x$.
- **Antisymmetry:** If $x \sqsubseteq y$ and $y \sqsubseteq x$ then $\alpha(\{x, y\}) = x$ and $\alpha(\{x, y\}) = y$ so $x = y$.
- **Transitivity:** Assume $x \sqsubseteq y$ and $y \sqsubseteq z$, i.e. $\alpha(\{x, y\}) = y$ and $\alpha(\{y, z\}) = z$. Let us compute

\[
\alpha \circ \alpha(S([x, y], \{y, z\})) = \alpha(\alpha([x, y], \alpha(\{y, z\}))) = \alpha(\{y, z\}) = z
\]

\[
= \alpha \circ \nu_X([x, y], \{y, z\}) = \alpha([x, y, z]).
\]

So

\[
\alpha([x, y, z]) = z. \quad (7)
\]

Let us compute

\[
\alpha \circ \nu_X([x, \{y, z\}]) = \alpha([x, y, z]) \overset{(7)}{=} z
\]

\[
= \alpha \circ \alpha(\alpha([x, \{y, z\}])) = \alpha(\alpha(\{x\}), \alpha(\{y, z\})) = \alpha([x, z]).
\]

So $\alpha([x, z]) = z$, therefore $x \sqsubseteq z$.

So $\sqsubseteq$ defines indeed an order on $X$. Now let us prove that for $A \in SX$, $\alpha(A)$ is the supremum of $A$. 

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• Upper bound: let \( a \in A \). We have

\[
\alpha \circ S \alpha (\{ a \}) = \alpha (\{ a \}) = \alpha (\{ a, \alpha (A) \})
\]

\[
= \alpha \circ \nu_X (\{ a \}) = \alpha (A \cup \{ a \}) = \alpha (A).
\]

So \( \alpha (\{ a, \alpha (A) \}) = \alpha (A), \) i.e. \( a \subseteq \alpha (A). \)

• Least upper bound: Assume there exists \( x \in X \) such that for all \( a \in A \) we have \( a \subseteq x \). Let us prove \( \alpha (A) \subseteq x \). By Lemma 3.15 we know that \( \{ [x, a] \mid a \in A \} \in SS(X, \leq) \). Let us compute

\[
\alpha \circ S \alpha (\{ [x, a] \mid a \in A \}) = \alpha (\{ [x, a] \mid a \in A \}) = \{ x \} = x
\]

\[
= \alpha \circ \nu_X (\{ [x, a] \mid a \in A \}) = \alpha (x \cup A).
\]

So

\[
\alpha (\{ x \} \cup A) = x.
\]

Let us compute

\[
\alpha \circ S \alpha (\{ x \}, A) = \alpha (\{ x \}, \alpha (A)) = \alpha (x, A)
\]

\[
\alpha \circ \nu_X (\{ x \} \cup A) = \alpha (\{ x \} \cup A) \overset{(8)}{=} x.
\]

So \( \alpha (\{ x, \alpha (A) \}) = x \), therefore \( \alpha (A) \subseteq x \).

Now if \( u \leq v \), by Proposition 3.14 \( \{ u \} \leq \{ u, v \} \leq \{ v \} \). Since \( \alpha \) preserves the preorder, we have \( \alpha (\{ u \}) \leq \alpha (\{ u, v \}) \leq \alpha (\{ v \}) \), so \( u \leq \alpha (\{ u, v \}) \leq v \).

This proof is inspired from [Mer10] 4.40.

With this Lemma we deduce the following.

**Lemma 3.17**

A morphism of \( S \)-algebras is a map that preserves the underlying preorder and the suprema with respect to \( \subseteq \) of non-empty compact subsets.

**Remark 3.18**

In the case of a free \( S \)-algebra, the order \( \subseteq \) is the inclusion.

**Lemma 3.19**

Let \( (X, \leq) \) be a preordered set. Assume \( X \) is equipped with an order \( \subseteq \) such that every non-empty compact subset \( A \) (for \( \tau_{\leq} \)) admits a supremum \( \bigvee (A) \) in the sense of \( \subseteq \). Assume moreover that the map \( \bigvee : S(X, \leq) \rightarrow (X, \leq) \) preserves the preorder relation. In this case the pair \( ((X, \leq), \bigvee) \) is an algebra on the monad \( S \).

**Proof.** We just need to check the two axioms of \( S \)-algebras.

• Let \( x \in X \). \( \bigvee \circ \delta_X (x) = \bigvee ([x]) = x \). So \( \bigvee \circ \delta_X = 1_X \).

• Let \( \mathcal{A} \in SS(X, \leq) \). \( \bigvee \circ S \bigvee (\mathcal{A}) = \bigvee ([\bigvee (A)] \mid A \in \mathcal{A}) \) and \( \bigvee \circ \nu_X (\mathcal{A}) = \bigvee (\bigcup_{A \in \mathcal{A}} A) \). By the properties of a supremum these two values are the same. Therefore \( \bigvee \circ S \bigvee = \bigvee \circ \nu_X \).

\( \square \)

This proof is inspired form [Mer10] 4.41.

A natural question that occurs when we have a monad is the following. Is the comparison functor \( \text{Compar} \) an equivalence?

\[
\text{Top}^K \xrightarrow{\text{Compar}} \text{Preord}^S \xleftarrow{\text{Preord}}.
\]

The functor \( \text{Compar} \) is defined on an object \( (X, \tau, \alpha) \) by \( \text{Compar}((X, \tau, \alpha)) = ((X, \leq_\tau, \alpha)_{|\mathcal{S}X}) \). The set \( \mathcal{S}X \) is the set of non-empty compact subsets for the topology \( \tau_{\leq_\tau} \). Since \( \tau \subset \tau_{\leq_\tau} \), \( \alpha_{|\mathcal{S}X} \) makes sense.

We prove that \( \text{Compar} \) is not an equivalence giving two objects in \( \text{Top}^K \) which are not isomorphic but which have the same image by \( \text{Compar} \). Let us consider the interval \( [0, 1] \) of real numbers with the usual topology \( \tau_\leq \) and the usual supremum \( \bigvee \).

The pair \( ((X, \tau_\leq), \bigvee) \) is an element of \( \text{Top}^K \) (see [Mer10] Example 4.44.2). Its image by \( \text{Compar} \) is \( ([0, 1], \leq_d, \bigvee_{|\mathcal{Pfin}[0, 1]}) \) where \( \mathcal{Pfin}[0, 1] \) denotes the family of finite subsets of \( [0, 1] \) and \( \leq_d \) is the discrete preorder, i.e. there are no distinct elements that are comparable. Now let us consider the interval \( [0, 1] \) of real numbers with the discrete topology \( \tau_d \) and with the usual supremum \( \bigvee \). The pair \( ((X, \tau_d), \bigvee) \) is an element of \( \text{Top}^K \) which is clearly not isomorphic to \( ((X, \tau_\leq), \bigvee) \). But \( \text{Compar}((X, \tau_\leq), \bigvee) \) is also equal to \( ([0, 1], \leq_d, \bigvee_{|\mathcal{Pfin}[0, 1]}) \).
3.2 Complete lattices as algebra over Set

It is well known (see for example [Bor4] chapter 4) that the category $\text{Sup}$ of complete lattices is isomorphic to the category of algebras on the monad $\mathbb{P}$ on $\text{Set}$ (the algebra structure is given by suprema). We write $\bigvee$ for the suprema. When we say morphism, this means a map that preserves arbitrary suprema. The category $\text{Set}$ is a monoidal category with the usual product and the monad $\mathbb{P}$ is compatible with the monoidal structure, the maps $\kappa_{A,B} : \mathcal{P}A \times \mathcal{P}B \to \mathcal{P}(A \times B)$ are given by $(A', B') \mapsto A' \times B'$ for $A$ and $B$ two sets. So we can (if it exists) consider the tensor product of two complete lattices (seen as $\mathbb{P}$-algebras). By Proposition 3.4 in [BW05] we know that it exists. The goal here is to describe this tensor product. We first show that given two complete lattices $A$ and $B$, the set $\text{Sup}(A, B)$ of maps from $A$ to $B$ preserving arbitrary suprema can itself be equipped with a complete lattice structure. Indeed for $F \subseteq \text{Sup}(A, B)$ we define $\bigvee(F) = \bigvee\{ f(a) \mid f \in F \}$. One can check that $\bigvee(F)$ is indeed in $\text{Sup}(A, B)$ and that this defines a complete lattice structure on $\text{Sup}(A, B)$.

If $A$, $B$ are two complete lattice, $A \times B$ is also a complete lattice with $\bigvee V := (\bigvee \pi_A(V), \bigvee \pi_B(V))$, for $V \subseteq A \times B$.

We also remark that in the case of complete lattices, a bimorphism $f : A \times B \to C$ is a map that preserves suprema in each variable.

Remark 3.20

Let $A$, $B$ and $C$ be three complete lattices. A morphisms $f : A \to \text{Sup}(B, C)$ is equivalent to a bimorphism $A \times B \to C$.

Notation

We write $\mathbb{T}$ for the complete lattice with the two elements $0$ and $1$, with $0 \leq 1$.

Lemma 3.21

Let $A, B$ be two complete lattices. The tensor product of $A$ and $B$ is given by

$$A \boxtimes B = \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T})), \mathbb{T}).$$

The bimorphism $q : A \times B \to \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T})), \mathbb{T})$ is given by $q(a, b) = \tau_{a,b}$, where $\tau_{a,b}(h) = h(a)(b)$, for $h \in \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T}))).$

Proof. • Let us prove that $q$ is a bimorphism. It is sufficient to prove that it preserves suprema in $A$ and in $B$. Let $h \in \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T}))), A' \subseteq A$ and $B' \subseteq B$. We have

$$q(\bigvee_{\alpha' \in A', \beta' \in B'} h(\alpha')(\beta'))(h) = \bigvee_{\alpha' \in A', \beta' \in B'} h(\alpha')(\beta')(h) = \bigvee_{\alpha' \in A', \beta' \in B'} q(\alpha'(\beta'))(h).$$

• Universal property: Let $C$ be a complete lattice and let $f : A \times B \to C$ be a bimorphism. We must show that there exists a unique morphism $f : \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T})), \mathbb{T}) \to C$ such that $f \cdot q = f$.

\[\begin{array}{cc}
A \times B & \xrightarrow{f} C \\
q \downarrow & \downarrow i \\
\text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T})), \mathbb{T}) & \xrightarrow{} C.
\end{array}\]

We first shown that for each morphism $\alpha \in \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T})), \mathbb{T})$ there exists a set $W_{\alpha} \subseteq A \times B$ such that $\alpha = \bigvee_{(x,y) \in W_{\alpha}} \tau_{x,y}$. We define

$$W_{\alpha} = \{(a, b) \in A \times B \mid \forall h \in \alpha^{-1}(0), h(a)(b) = 0\}.$$ 

Proving $\alpha = \bigvee_{(x,y) \in W_{\alpha}} \tau_{x,y}$ is equivalent to proving that for each $h \in \text{Sup}(\text{Sup}(A, \text{Sup}(B, \mathbb{T})))$ we have

$$\alpha(h) = 0 \iff \forall (x,y) \in W_{\alpha}, h(x)(y) = 0.$$

The implication from left to right is clear. To see the converse, let us prove that $h \leq \bigvee \alpha^{-1}(0)$. If $(x, y) \in W_{\alpha}$ we have $h(x)(y) = 0$ and $\bigvee \alpha^{-1}(0)(x)(y) = 0$. If $(x, y) \notin W_{\alpha}$, we have $\bigvee \alpha^{-1}(0)(x)(y) = 1$. So in all cases, we have $h(x)(y) \leq \bigvee \alpha^{-1}(0)(x)(y)$, so $h \leq \bigvee \alpha^{-1}(0)$. By monotonicity of $\alpha$, we have $\alpha(h) = 0$.

Now we claim that if there exists a set $W \subseteq A \times B$ such that $\alpha = \bigvee_{(x,y) \in W} \tau_{x,y}$ then for all $(x, y) \in W$ the following happens: either $x = 0$ or $y = 0$, either there exists $(a, b) \in W_{\alpha}$ such that $x \leq a$ and $y \leq b$. To see this, let us first remark that since $\alpha$ preserves arbitrary suprema, $W_{\alpha}$ has the following properties.

$$(a', b') \leq (a, b) \in W_{\alpha} \implies (a', b') \in W_{\alpha};$$

13
(0, 1), (1, 0) \in W\alpha;
(a_i, b_i)_{i \in I} \subset W\alpha \implies \bigvee_{i \in I} a_i, b_i \in W\alpha;
(a, b_j)_{j \in J} \subset W\alpha \implies (a, \bigvee_{j \in J} b_j) \in W\alpha.

With this one checks that the map \(h_{W\alpha}\) defined by

\[
h_{W\alpha}(x)(y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \text{ or } \exists (a, b) \in W\alpha \text{ s.t. } (x, y) \leq (a, b) \\ 1 & \text{else} \end{cases}
\]

is in \(\text{Sup}(A, \text{Sup}(B, \mathbb{2}))\). This map satisfies \(\alpha(h_{W\alpha}) = 0\). This proves the claim.

Now to satisfy the fact that \(f\) preserves suprema, we must have \(f(V(A, y)) = V(\tau x, y)\), for all \(W \subset A \times B\).

To satisfy the commutativity of the diagram, we must have \(f(\alpha) = f(a, b)\). So, for \(\alpha \in \text{Sup}(A, \text{Sup}(B, \mathbb{2}))\), we define \(f(\alpha) := V(\tau x, y)\), with \(W\alpha\) defined as above.

To complete the proof, we just need to show that if \(\alpha = V(\tau x, y)\) then \(V(\tau x, y) = V(\tau x, y)\), but this is consequence of the claim, since \(f\) is a bimorphism.

**Remarks 3.22**

In [Shm74] the tensor products of two complete lattices \(A\) and \(B\) is construct as \(\text{Sup}(A, B^{\text{opp}})\).

In [KW10] the tensor products of two complete lattices \(A\) and \(B\) is construct as \(\mathcal{P}_{\text{Sup}}(A, B)\), where

\[
\mathcal{P}_{\text{Sup}}(A, B) = \left\{ V \in \mathcal{P}(A \times B) \text{ s.t. } (x, y) \leq (a, b) \in V \implies (x, y) \in V \forall (S, T) \in \mathcal{P}_0 A \times \mathcal{P}_0 B, S \times T \subset V \implies (V, V) \in V \right\}.
\]

In fact an element of \(\mathcal{P}_{\text{Sup}}(A, B)\) corresponds to the set \(W\alpha\) in the proof of Lemma 3.21.

### 3.3 Almost complete sup-semilattices

**Definition**

An **almost complete sup-semilattice** is a poset for which every non-empty subset admits a supremum. A **morphism of almost complete sup-semilattices** is a map that preserves all non-empty suprema.

The category of almost complete sup-semilattices is denoted by \(\text{Sup}_a\). In a similar way as proving that the category of complete lattices is isomorphic to the category \(\text{Set}^\mathbb{P}\), we prove that the category \(\text{Sup}_a\) is isomorphic to the category \(\text{Set}^\mathbb{P}_0\), where \(\mathbb{P}_0\) is the monad as \(\mathbb{P}\) but which take only the non-empty subset. As in section 3.2, the monad \(\mathbb{P}_0\) is monoidal on the monoidal category \(\text{Set}^{\times}\). The maps \(\kappa_{A,B}\) are the same as for complete lattices but restricted to non-empty subsets. In this case, a bimorphism is a map that preserves non-empty suprema in each variable.

As for complete lattices, if \(A, B\) are two almost complete sup-semilattices, \(A \times B\) is an almost complete sup-semilattice by setting \(V := (\bigvee \pi_1(V), \bigvee \pi_2(V))\), for \(0 \neq V \subset A \times B\).

**Definition**

For \(A, B\) two almost complete sup-semilattices, we define \(\mathcal{P}_{\text{Sup}_a}(A, B)\) as follows:

\[
\mathcal{P}_{\text{Sup}_a}(A, B) := \left\{ V \in \mathcal{P}_0 (A \times B) \text{ s.t. } (x, y) \leq (a, b) \in V \implies (x, y) \in V \forall (S, T) \in \mathcal{P}_0 A \times \mathcal{P}_0 B, S \times T \subset V \implies (V, V) \in V \right\}.
\]

Ordering the set \(\mathcal{P}_{\text{Sup}_a}(A, B)\) by inclusion, one can verify that it is an almost complete lattice. The infima of a subset of \(\mathcal{P}_{\text{Sup}_a}(A, B)\) is given by the intersection. Such a set is considered in [KW10] but for complete lattices. If \(a \in A\) and \(b \in B\) then \(\downarrow (a, b) = \downarrow a \times \downarrow b \in \mathcal{P}_{\text{Sup}_a}(A, B)\). We denote the element \(\downarrow (a, b)\) by \(a \otimes b\).

**Example 3.23**

Let us consider the following almost complete sup-semilattices:

\[
A = \begin{array}{c}
x \downarrow \\
\uparrow z \\
y \uparrow \\
\end{array}
B = \begin{array}{c}
a \downarrow \\
\uparrow c \\
b \uparrow \\
\end{array}
\]

Then we have

\[
\mathcal{P}_{\text{Sup}_a}(A, B) = \left\{ \begin{array}{l}
\{ (x, a) \}, \{ (x, a), (y, a), (z, a) \}, \{ (x, a), (z, a), (y, a), (x, c), (x, b) \} \\
\{ (y, b) \}, \{ (x, b), (y, b), (z, b) \}, \{ (x, b), (y, b), (z, b), (x, a), (x, c) \} \\
\{ (x, b) \}, \{ (x, a), (x, b), (x, c) \}, \{ (x, a), (z, a), (y, a), (y, c), (y, b) \} \\
\{ (y, a) \}, \{ (y, a), (y, b), (y, c) \}, \{ (x, b), (y, b), (z, b), (y, a), (y, c) \} \\
\{ (x, a), (y, b) \}, \{ (x, b), (y, a) \}, \{ (x, a), (y, b), (z, b), (y, a), (y, c) \} \\
\end{array} \right\}
A \times B
\]
Now, let us compute $\text{Sup}_0(\text{Sup}_0(A, \text{Sup}_0(B, \mathfrak{F})), \mathfrak{F})$. We write an application by its graph and we represent a poset by it Hasse diagram. For $\text{Sup}_0(B, \mathfrak{F})$, we have

\begin{align*}
\{(x,1), (y,1), (z,1)\} & := \alpha \\
\{(x,0), (y,1), (z,1)\} & := \beta \\
\{(x,1), (y,0), (z,1)\} & := \gamma \\
\{(x,0), (y,0), (z,0)\} & := \delta .
\end{align*}
For \( \text{Sup}_0(A, \text{Sup}_0(B, \bar{2})) \), we have

\[
\{(x, \alpha), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \beta), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \gamma), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \delta), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \gamma), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \delta), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \beta), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \gamma), (y, \alpha), (z, \alpha)\}
\]

\[
\{(x, \delta), (y, \alpha), (z, \alpha)\}
\]

We do not draw \( \text{Sup}_0( \text{Sup}_0( A, \text{Sup}_0(B, \bar{2})), \bar{2}) \) but we just say that it is as \( \text{Sup}_0(A, \text{Sup}_0(B, \bar{2})) \) but with the inverse order and with a new element over all the others (the element corresponding to the map that sends all on \( 1 \)). This example proves that in the case of almost complete sup-semilattices we do not have a bijection between \( \mathcal{R}_{\text{Sup}_0}(A, B) \) and \( \text{Sup}_0(\text{Sup}_0(A, \text{Sup}_0(B, \bar{2})), \bar{2}) \).

We do not have an explicit description of the tensor product of two almost complete sup-semilattices (seen as \( \mathbb{P}_0 \) algebras), but we know that it exists since \( \text{Set}^{\mathbb{P}_0} \) is cocomplete (Proposition 3.4 in \([BW05]\)).

### 3.4 Continuous lattices

**Definition**

Let \((X, \leq)\) be a complete lattice and let \( x, y \in X \). We say that \( x \) is way below \( y \) and we write \( x \ll y \) if whenever \( S \) is a up-directed set and \( y \leq \bigvee S \), there exists \( s \in S \) such that \( x \leq s \).

From this definition follow some elementary properties.

**Proposition 3.24**

i) If \( x \ll y \) then \( x \leq y \).

ii) \( 0 \ll x \) for all \( x \in X \).

iii) If \( x \ll y \) and \( w \leq x \) and \( y \leq z \) then \( w \ll y \) and \( x \ll z \).

**Remark 3.25**

In particular every \( y \in X \) satisfies \( y \geq \bigvee \{x \in X \mid x \ll y\} \).

**Example 3.26**

In the complete lattice \(([0, 1], \leq)\), the relation \( \ll \) means \( < \) except for \((0, 0)\) for which we also have \( 0 \ll 0 \).

**Definition**

We say that a complete lattice \((X, \leq)\) is a **continuous** lattice if for every \( y \in X \), we have

\[
y = \bigvee \{x \in X \mid x \ll y\}.
\]

A **morphism of continuous lattices** is a map preserving all infima and all up-directed suprema.

The category of continuous lattices is denoted by \( \text{Cnt} \).
Example 3.27
Every finite lattice is a continuous lattice, since if $S$ is an up-directed set, then $\bigvee S \in S$.

Example 3.28
The complete lattice $([0,1], \leq)$ is a continuous lattice.

In [Joh82, Section VII], there are good descriptions of continuous lattices.

It is proved in [Wyl81, Section 8] that the category of continuous lattices is isomorphic to the category $\text{Set}^\mathbb{F}$, where $\mathbb{F}$ is the filter monad. The monad $\mathbb{F}$ is monoidal on $\text{(Set, \times)}$. The maps $\kappa_{A,B} : \mathcal{F}A \times \mathcal{F}B \to \mathcal{F}(A \times B)$ are defined by $\kappa_{A,B}(\xi_1, \xi_2) = \{V \subset A \times B \mid \exists (A, B) \in \xi_1 \times \xi_2 \text{ st. } A \times B \subset V\}$. So, if we see two continuous lattices as elements of $\text{Set}^\mathbb{F}$ we can consider the tensor product. This tensor product exists, since $\text{Set}^\mathbb{F}$ is cocomplete (Proposition 3.4 in [BW05]). Unfortunately, we do not have an explicit description of this tensor product.

3.4.1 The Scott topology

We define a functor $\text{TopolScott} : \text{Cnt} \to \text{T}_0\text{Top}$ as follows. For $(X, \leq)$ a complete lattice we define $\text{TopolScott}(X, \leq) = (X, \tau_{\leq,S})$, where $\tau_{\leq,S}$ is the family of subsets $U$ of $X$ satisfying,

1. if $x \in U$ and $x \leq y$ then $y \in U$,
2. if $S$ is an up-directed set and $\bigvee S \in U$ then $S \cap U \neq \emptyset$.

One checks easily that this defines a topology. To see that this topology is $\text{T}_0$ let us remark that if $x \nleq y$ the set $\{z \in X \mid z \nleq y\}$ is open, contains $x$ and does not contain $y$.

If $f : (X, \leq) \to (X', \leq')$ is a morphism of continuous lattices, then we define $\text{TopolScott}(f) = f$. Let us prove that in this case $f$ is continuous $(X, \tau_{\leq,S}) \to (X', \tau_{\leq',S})$. Let $U \in \tau_{\leq,S}$ and let us prove that $f^{-1}(U) \in \tau_{\leq,S}$. If $x \in f^{-1}(U)$ and $x \leq y$ then $f(x) \leq f(y)$, since $f$ is monotone, so $y \in f^{-1}(U)$. Let $S$ be an up-directed set such that $\bigvee S \in f^{-1}(U)$. Since $f$ preserves up-directed suprema, we have $f(\bigvee S) = \bigvee \{f(s) \mid s \in S\} \in U$. So, since $U \in \tau_{\leq,S}$, there exists $s \in S$ such that $f(s) \in U$. Therefore $f^{-1}(U) \in \tau_{\leq,S}$.

This topology comes from [Sco72].

Remark 3.29
With this topology, the definition of $x \ll y$ is equivalent to $y \in \text{Int}\{z \in X \mid x \leq z\}$, where the interior is in the sense of $\tau_{\leq,S}$.

Remark 3.30
If $(X, \leq)$ is a continuous lattice then $(X, \leq) = (X, \leq_{\tau_{\leq,S}})$. Indeed if $x \leq y$ and $U \in \tau_{\leq,S}$ satisfy $x \in U$, then $y \in U$. So $x \leq_{\tau_{\leq,S}} y$. Conversely, if $x \nleq y$ then the open set $\{z \in X \mid z \nleq y\}$ contains $x$ but does not contain $y$. So $x \nleq_{\tau_{\leq,S}} y$.

3.5 $\mathbb{K}$-algebras on the category of compact and Hausdorff topological spaces

In Appendix A.1, it is mentioned that we have a monad $\mathbb{K}$ on the category of the compact and Hausdorff topological spaces. A $\mathbb{K}$-algebra $(X, \alpha)$ gives rise to an almost complete sup-semilattice with the order given by $x \leq y$ if and only if $\alpha([x,y]) = y$, $\bigvee X' = \alpha(\mathbb{X})$. In this section, we study this $\mathbb{K}$-algebras. In this section, every topological space is supposed to be compact and Hausdorff and if it has a $\mathbb{K}$-algebra structure we write it by $\bigvee$ and by $\leq$ the corresponding order.

Proposition 3.31
Let $(X, \bigvee)$ be a $\mathbb{K}$-algebra and $y \in X$. The map $\phi_y : X \to X$ defined by $\phi_y(x) = \bigvee \{x, y\}$ is continuous.

Proof. The map $\phi_y$ is the composition of the following maps:

$$
\begin{array}{ccc}
X & \longrightarrow & X \times X \\
  & \downarrow \phi_y & \downarrow \\mathcal{F}X \\
  & \longrightarrow & \mathcal{F}X \\
  & \downarrow \phi_y & \downarrow \mathcal{F}X \\
  & \longrightarrow & X \\
\end{array}
$$

The first is continuous by properties of the product topology. The second is continuous by Lemma 4.2 [Mer10]. The third is continuous by definition of a $\mathbb{K}$-algebra. So $\phi_y$ is continuous as composition of continuous maps.
Corollary 3.32
Let \((X,\mathcal{V})\) be a \(K\)-algebra and \(y \in X\). The set \(\downarrow y = \{x \in X \mid x \leq y\}\) is closed.

Proof. The set \(\downarrow y\) is the preimage of the closed set \(\{y\}\) \((X\) is Hausdorff\) by the continuous map \(\phi_y\).

Corollary 3.33
Let \((X,\mathcal{V})\) be a \(K\)-algebra and \(U \subset X\) an open subset. The set \(\downarrow U = \{x \in X \mid \exists u \in U\text{ s.t. } x \leq u\}\) is open.

Proof. One can write
\[
\downarrow U = \bigcup_{u \in U} \{x \in X \mid \{x,u\} \in U\} = \bigcup_{u \in U} \phi_u^{-1}(U).
\]
Every set appearing in the union is open as the preimage of an open set by a continuous map. So \(\downarrow U\) is open.

If \(A\) and \(B\) are two \(K\)-algebras then \(A \times B\) can also be equipped with a \(K\)-algebra structure in the same way as for almost complete sup-semilattice. If \(V \in \mathcal{A}(A \times B)\), we define \(\mathcal{V} := (\{\pi_A(V)\}, \{\pi_B(V)\})\). This suprema map: \(\mathcal{A}(A \times B) \to A \times B\) is continuous because it is the composition of the following maps which are all continuous.

The category \(\text{CompHaus}\) is monoidal with the Cartesian product. This can be proved directly or by Paragraph 2.1. A neutral element is a topological space with only one element, which is clearly compact and Hausdorff. Moreover the monad \(K\) is monoidal on \(\text{CompHaus}\). Indeed, as for the monad \(P\) on \(\text{Set}\), the maps \(\kappa_{A,B} : (A', B') \to A' \times B'\) makes commute all the diagrams in the definition 2.2. The only thing to prove is that \(\kappa_{A,B}\) is indeed a morphism in \(\text{CompHaus}\) if \(A\) and \(B\) are such objects, but this is true by Proposition A.1. So we can define bi-morphisms in the category \(\text{CompHaus}^K\). A bimorphism is a continuous map with preserves non-empty compact supremum in each variable. If the coequalizer in definition 2.3 exists in \(\text{CompHaus}^K\), we can define the tensor product of two \(K\)-algebras.

In fact it is proved in [Wyl85] Theorem 3.4 that the category \(\text{CompHaus}^K\) is isomorphic to the category \(\text{Set}^\mathbb{N}_0\) which is complete by Proposition 3.4 in [BW05]. Therefore the tensor product of two \(K\)-algebras exists. Unfortunately, we do not have an explicit description of it.

4 Ultraspaces

Given a topological space \((X,\tau)\), we can construct the topological space \((\mathcal{X}X,\tau^1)\) (see Appendix A.1). Since \((\mathcal{X}X,\tau^1)\) is still a topological space, we can again apply the functor \(K\) to define \((\mathcal{X}^\alpha X,\tau^\alpha)\) for every ordinal \(\alpha\). The notations used here for ordinals are explain in Appendix A.2. In this section, the greek letters denote ordinals. We define \((\mathcal{X}^\alpha X,\tau^\alpha)\) for \(\alpha = 0\), for \(\alpha = \beta + 1\) (successor) and for \(\alpha = \bigcup_{\beta<\alpha} \beta\) (limit). Before doing this, let us defined it for \(\omega\). For \(n, k \in \mathbb{N}\), we have the continuous map \(\{\cdot\}^k_{\omega} : \mathcal{X}^n X \to \mathcal{X}^{n+k} X\) which is the composition of the maps \(\{\cdot\}^j_{\omega} \circ \mathcal{X} X\) for \(j \in \{0,\cdots,k - 1\}\). Now we define \((\mathcal{X}^\alpha X, \tau^\alpha)\) by

\[
\mathcal{X}^\alpha X = \bigcup_{n \in \mathbb{N}} \mathcal{X}^n X / \sim,
\]
where \(\sim\) is the equivalence relation defined as follows. If \(x \in \mathcal{X}^n X\) and \(y \in \mathcal{X}^{n+k} X\) with \(n, k \in \mathbb{N}\), we define \(x \sim y\) \(\iff\) \(\{x\}_{\omega} = y\). For every \(\mathcal{X}^n X\), \(n \in \mathbb{N}\), we have the map \(\{\cdot\}^n_{\omega} : \mathcal{X}^n X \to \mathcal{X}^\omega X\) which sends an element to its equivalence class. We define \(\tau^\omega\) to be the finest topology on \(\mathcal{X}^\omega X\) such that every map \(\{\cdot\}^n_{\omega} \circ \mathcal{X} X\) is continuous. This means that a subset of \(\mathcal{X}^\omega X\) is open for \(\tau^\omega\) if and only if its preimage by every map \(\{\cdot\}^n_{\omega} \circ \mathcal{X} X\) is open.

Now let us define \((\mathcal{X}^\alpha X, \tau^\alpha)\) for a general ordinal \(\alpha\).

1. For \(\alpha = 0\), we define \((\mathcal{X}^0 X, \tau^0) := (X, \tau)\). And we define the continuous map \(\{\cdot\}^0_{X} = 1_{X}\).

2. For \(\alpha = \beta + 1\), we define \((\mathcal{X}^{\beta+1} X, \tau^{\beta+1}) := (\mathcal{X}^\beta X, \tau^{\beta+1})\) as explained in Appendix A.1. For every ordinal \(\gamma < \beta\), we define the continuous map \(\{\cdot\}^{\delta}_{\omega} \times \mathcal{X} X \to \mathcal{X}^{\beta+1} X\) by \(\{\cdot\}^{\delta}_{\omega} \times \mathcal{X} X\) where \(\delta\) is the ordinal such that \(\gamma + \delta = \beta\) and assuming the continuous maps \(\{\cdot\}^{\delta}_{\omega} \times \mathcal{X} X \to \mathcal{X}^\omega X\) are already defined.

3. For \(\alpha = \bigcup_{\beta<\alpha} \beta\), \(\mathcal{X}^\alpha X\) is defined by

\[
\mathcal{X}^\alpha X = \bigcup_{\beta<\alpha} \mathcal{X}^\beta X / \sim,
\]
where \(\sim\) is the equivalence relation defined as follows. For \(x \in \mathcal{X}^\beta X\) and \(y \in \mathcal{X}^{\beta+1} X\) we have \(x \sim y\) if on only if there exists an ordinal \(\delta < \alpha\) such that \(\beta_1 + \delta = \beta_2\) and \(y = \{\cdot\}^{\delta}_{\omega} \times \mathcal{X} X(x)\).
For each $\beta < \alpha$ we define the map $\gamma \mapsto \mathcal{X}^\beta X : \mathcal{X}^\beta X \to \mathcal{X}^\alpha X$ which sends an element $x$ to its equivalence class, for $\gamma$ the ordinal such that $\alpha = \beta + \gamma$.

We define $\tau^\alpha$ to be the finest topology on $\mathcal{X}^\alpha X$ such that every map $\gamma \mapsto \mathcal{X}^\beta X$ is continuous for $\beta < \alpha$. This means that a subset of $\mathcal{X}^\alpha X$ is open for $\tau^\alpha$ if and only if its preimage by every map $\gamma \mapsto \mathcal{X}^\beta X$ is open.

**Remarks 4.1**

For every ordinal $\alpha$ and every ordinal $\beta < \alpha$ the continuous map $\gamma \mapsto \mathcal{X}^\beta X : \mathcal{X}^\beta X \to \mathcal{X}^\alpha X$ is injective.

Let $\gamma < \beta < \alpha$ be ordinals and $\delta$ be the ordinal such that $\gamma + \delta = \beta$ and $\epsilon$ be the ordinal such that $\beta + \epsilon = \alpha$ and $U \in \tau^\beta$. Then $(\mathcal{X}^{\delta + \gamma} X)^{-1}(\mathcal{X}^\beta X(U)) \in \tau^\gamma$ because $(\mathcal{X}^{\delta + \gamma} X)^{-1}(\mathcal{X}^\beta X(U)) = (\mathcal{X}^\gamma X(U)).$

**Remark 4.2**

For a limit ordinal $\alpha$, the topology $\tau^\alpha$ is called the final topology. If we consider the category of ordinals $\beta < \alpha$ with morphisms the relation $\leq$, the space $(\mathcal{X}^\alpha X, \tau^\alpha)$ is the colimit in $\text{Top}$ of the functor assigning to an ordinal $\beta < \alpha$ the space $(\mathcal{X}^\beta X, \tau^\beta)$.

From now on, if we write $\gamma \mapsto \mathcal{X}^\beta X : \mathcal{X}^\beta X \to \mathcal{X}^\alpha X$, it is understood that $\gamma$ is the ordinal such that $\beta + \gamma = \alpha$.

The affirmations in Appendix A.1 which say that if $(X, \tau)$ is discrete so is $(\mathcal{X}^\alpha X, \tau^\beta)$ and that if $(X, \tau)$ is T$_2$ so is $(\mathcal{X}^\alpha X, \tau^\beta)$ and that if $(X, \tau)$ is compact so is $(\mathcal{X}^\alpha X, \tau^\beta)$, can be reformulated as follows. The discrete property and the T$_2$ property and the compact property are preserved by successor. What about limit ordinals?

**Proposition 4.3**

Let $(X, \tau)$ be a discrete topological space. Then $(\mathcal{X}^\alpha X, \tau^\alpha)$ is discrete for every ordinal $\alpha$.

**Proof.** We prove this by induction on $\alpha$. We just need to check that the discrete property is preserved by limit ordinal. This is true because, since every $(\mathcal{X}^\beta X, \tau^\beta)$ is discrete for $\beta < \alpha$, the maps $\gamma \mapsto \mathcal{X}^\beta X : \mathcal{X}^\beta X \to \mathcal{X}^\alpha X$ are all continuous for any topology on $\mathcal{X}^\alpha X$. So the discrete is the finest such that every map $\gamma \mapsto \mathcal{X}^\beta X$ is continuous.

Now we will see that the compact property is not preserved by limit ordinal in general.

**Example 4.4**

Let $(X, \tau)$ be a finite discrete topological space with at least two elements. The space $\mathcal{X}^\beta X$ is infinite because for every $n \in \mathbb{N}$ we have $|\mathcal{X}^{n+1} X| = 2^{|\mathcal{X}^n X|} - 1$, and every map $\gamma \mapsto \mathcal{X}^\beta X : \mathcal{X}^\beta X \to \mathcal{X}^\alpha X$ is injective. Since the set $(\mathcal{X}^\alpha X, \tau^\alpha)$ is discrete, it cannot be compact.

For the functor $K : \text{Top} \to \text{Top}$ which sends a topological space $(X, \tau)$ to $(\mathcal{X}^\alpha X, \tau^\alpha)$, one also defines $K^\alpha$ for every ordinal $\alpha$. On the objects it is already defined. Let us define it on the morphisms by induction on $\alpha$. Let $f : (X, \tau) \to (X', \tau')$ be a continuous map $(X \neq \emptyset)$ and let us define $K^\alpha f : (\mathcal{X}^\alpha X, \tau^\alpha) \to (\mathcal{X}^\alpha X', \tau'^\alpha)$, and let us prove also by induction the property $P(\alpha)$ which is: if $\alpha = \beta + \gamma$ then $K^\epsilon f \circ \gamma \mapsto \mathcal{X}^\beta X = \gamma \mapsto \mathcal{X}^\beta X \circ K^\gamma f$. 

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i) If $\alpha = 0$, we define $K^0 f := f$, which is continuous.

ii) If $\alpha = \beta + 1$ we define $K^{\beta+1} f := K(K^\beta f)$. The map $K^{\beta+1} f$ is continuous by the fact that $K$ is a functor and by the induction hypothesis.

Since $\{\} = \{\}$ is a natural transformation $\mathsf{I}_{\mathsf{Top}} \Rightarrow K$, we have $K^{\beta+1} f \circ \{\} = \{\} \circ K^\beta f$. The property $P(\alpha')$ now holds for every ordinal $\alpha' \leq \alpha$.

iii) If $\alpha = \bigcup_{\beta < \alpha} \beta$ we define $K \bigcup_{\beta < \alpha} \beta f$ as follows. Let $x \in \mathcal{K} \bigcup_{\beta < \alpha} \beta X = \bigcup_{\beta < \alpha} \mathcal{K}^{\beta} X/ \sim$. There exists $\beta < \alpha$ and $y \in \mathcal{K}^{\beta} X$ such that $y$ is a representative of $x$. We define $K^\alpha f(x) := \{\}^{\gamma} \mathcal{K}^{\beta} X, \circ K^\beta f(x)$, where $\gamma$ is the ordinal such that $\beta + \gamma = \alpha$. Let us prove that this is well-defined. Assume there exists $\delta < \alpha$ and $z \in \mathcal{K}^{\delta} X$ such that $z$ is also a representative of $x$. Let $\epsilon$ be the ordinal such that $\delta + \epsilon = \alpha$ and let us prove that $K^\delta f(y)$ is equivalent to $K^\delta f(z)$ in $\bigcup_{\beta < \alpha} \mathcal{K}^{\beta} X'$. Without loss of generality, we can assume that there exists an ordinal $\zeta$ such that $\beta + \zeta = \beta$. In this case we have $\{\}^{\gamma} \mathcal{K}^{\beta} X(z) = y$. By hypothesis $P(\beta)$, $K^\delta f(y) = K^\delta f \circ \{\}^{\gamma} \mathcal{K}^{\beta} X(z) = \{\}^{\gamma} \mathcal{K}^{\beta} X, \circ K^\delta f(z)$. This proves that $K^\delta f(y)$ is equivalent to $K^\delta f(z)$ in $\bigcup_{\beta < \alpha} \mathcal{K}^{\beta} X$ and therefore $K^\alpha f$ is well-defined.

Now let us prove that $K^\alpha f$ is continuous. Let $O \in \tau^\alpha$. To see that $(K^\alpha f)^{-1}(O) \in \tau^\alpha$ we must check that for all $\beta < \alpha$ and $\gamma$ such that $\beta + \gamma = \alpha$, we have $\{\}^{\gamma} \mathcal{K}^{\beta} X)^{-1}((K^\alpha f)^{-1}(O)) = (K^\delta f)^{-1}(U)$.

By definition of $O$, we have $\{\}^{\gamma} \mathcal{K}^{\beta} X)^{-1}((K^\alpha f)^{-1}(O)) = \{y \in \mathcal{K}^{\beta} X | K^\delta f(y) \in O\} = \{y \in \mathcal{K}^{\beta} X | K^\delta f(y) \in \{\}^{\gamma} \mathcal{K}^{\beta} X, \circ K^\delta f(z)\}$.

By induction hypothesis, $K^\delta f$ is continuous. So let us consider the case $\beta < \alpha$.

Again by induction on $\alpha$, one can prove that $K^\alpha$ agrees with composition of morphisms, so it is indeed a functor $\mathsf{Top} \to \mathsf{Top}$.

**Remark 4.5**

In this construction, we also have proved that for every ordinals $\alpha, \beta, \gamma$ such that $\beta + \gamma = \alpha$, $\{\}^{\gamma} \mathcal{K}^{\beta} X$ is a natural transformation $K^\beta \Rightarrow K^\alpha$.

Now it is time to give some open subsets of $K^\alpha(X, \tau)$ for $\alpha$ limit. We define the subset $W^\alpha(U) \subset \mathcal{K}^{\alpha} X$ by induction on $\alpha$, for $U \in \tau$.

i) $W^0(U) = U$;

ii) $W^{\beta+1}(U) = W(W^\beta(U))$;

iii) $W \bigcup_{\beta < \alpha} \beta(U) = \bigcup_{\beta < \alpha} \{\}^{\gamma} \mathcal{K}^{\beta} X(W^\beta)$.

**Proposition 4.6**

*For every ordinal $\beta < \alpha$, $\{\}^{\gamma} \mathcal{K}^{\beta} X(W^\beta(U)) \subset W^\alpha(U)$.*

**Proof.** By induction on $\alpha$.

i) If $\alpha = 0$, noting to do.

ii) If $\alpha = \beta' + 1$, we have by definition and induction hypothesis

$$
\{\}^{\gamma} \mathcal{K}^{\beta'} X(W^\beta(U)) = \{\}^{\gamma} \mathcal{K}^{\beta'} X(W^\beta(U)) \subset \{\}^{\gamma} \mathcal{K}^{\beta'} X(W^\beta(U)) \subset W^\alpha(U).
$$

iii) If $\alpha = \bigcup_{\beta < \alpha} \beta'$, $\{\}^{\gamma} \mathcal{K}^{\beta'} X(W^\beta(U)) \subset \bigcup_{\beta < \alpha} \{\}^{\gamma} \mathcal{K}^{\beta'} X(W^\beta) = W^\alpha(U)$.

**Lemma 4.7**

*Let $U \in \tau$ and $\alpha$ an ordinal. The set $W^\alpha(U)$ is open in $(\mathcal{K}^{\alpha} X, \tau^\alpha)$.*

**Proof.** By induction on $\alpha$. The case $\alpha = 0$ is by definition and the case $\alpha = \beta + 1$ is also by definition if we assume that it is true for $\beta$. So let us consider the case $\alpha = \bigcup_{\beta < \alpha} \beta$. We must check that, for every $\beta' < \alpha$, $\{\}^{\gamma} \mathcal{K}^{\beta'} X)^{-1}(W^\alpha) \in \tau^\beta'$. We have

$$
\{\}^{\gamma} \mathcal{K}^{\beta'} X)^{-1}(\bigcup_{\beta < \alpha} \{\}^{\gamma} \mathcal{K}^{\beta} X(W^\beta)) = \{x \in \mathcal{K}^{\beta'} X | \exists \beta < \alpha, \exists y \in W^\beta U \text{ s.t. } x \sim y\}
$$
Let us consider the case to find the tensor product of elements of $\text{CompHaus}$. In a similarly way as the proof of Lemmas 4.7 and 4.8 one has the following result.

**Proposition 4.6.** Let $(X, \tau)$ be a Hausdorff topological space. Then $(\mathcal{X}^\alpha X, \tau^\alpha)$ is Hausdorff for every ordinal $\alpha$.

**Proof.** By induction on $\alpha$. The case $\alpha = 0$ is by definition. The case successor is clear by definition of the $W(\cdot)$. So let us consider the case $\alpha = \bigcup_{\beta<\alpha} \beta$ assuming $W^\beta(\mathcal{U}) \cap W^\beta(\mathcal{V}) = \emptyset$ for every $\beta < \alpha$. Suppose by contradiction that there exists $x \in W^\alpha(\mathcal{U}) \cap W^\alpha(\mathcal{V})$. This means that there exists $\beta_1 < \alpha$, $\beta_2 < \alpha$, $y_1 \in W^{\beta_1}(\mathcal{U})$ and $y_2 \in W^{\beta_2}(\mathcal{V})$ such that $x = \{\}_{\mathcal{X}^\beta X}(y_1) = \{\}_{\mathcal{X}^\beta X}(y_2)$. Without loss of generality, we can assume $\beta_1 = \beta_2 + \delta$. So we have $\{\}_{\mathcal{X}^{\beta_1} X}(y_1) = \{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X} \cap \{\}^{\beta_2}_{\mathcal{X}^{\beta_2} X}(y_2)$. This implies that $y_1 = \{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X}(y_2)$. By Proposition 4.6 $\{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X}(y_2) \in W^{\beta_1}(\mathcal{V})$. This contradicts $W^{\beta_1}(\mathcal{U}) \cap W^{\beta_1}(\mathcal{V}) = \emptyset$.

Now we will consider this after $\tau$. Let $\beta < \alpha$ and $U \in \tau^\beta$. We define $W^\beta_\alpha(U) \subset \mathcal{X}^\alpha X$ by induction on $\alpha > \beta$:

1. $W^\beta_\alpha(U) := U$.
2. $W^{\beta+1}_\alpha(U) := W(W^\beta_\alpha(U))$.
3. $W_{\beta \leq \gamma \leq \alpha} = \bigcup_{\beta \leq \gamma < \alpha} \{\}^{\gamma}_{\mathcal{X}^\gamma X}(W^\beta_\alpha(U))$.

In a similarly way as the proof of Lemmas 4.7 and 4.8 one has the following result.

**Lemma 4.8.** If $U, V \in \tau$ are disjoint, then $W^\alpha_\tau(U)$ and $W^\alpha_\tau(V)$ are also disjoint, for any ordinal $\alpha$.

**Proof.** By induction on $\alpha$. The case $\alpha = 0$ is by definition. The case successor is clear by definition of the $W(\cdot)$. So let us consider the case $\alpha = \bigcup_{\beta<\alpha} \beta$ assuming $W^\beta_\alpha(U) \cap W^\beta_\alpha(V) = \emptyset$ for every $\beta < \alpha$. Suppose by contradiction that there exists $x \in W^\alpha_\tau(U) \cap W^\alpha_\tau(V)$. This means that there exists $\beta_1 < \alpha$, $\beta_2 < \alpha$, $y_1 \in W^{\beta_1}_\alpha(U)$ and $y_2 \in W^{\beta_2}_\alpha(V)$ such that $x = \{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X}(y_1) = \{\}^{\beta_2}_{\mathcal{X}^{\beta_2} X}(y_2)$. Without loss of generality, we can assume $\beta_1 = \beta_2 + \delta$. So we have $\{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X}(y_1) = \{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X} \cap \{\}^{\beta_2}_{\mathcal{X}^{\beta_2} X}(y_2)$. This implies that $y_1 = \{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X}(y_2)$. By Proposition 4.6 $\{\}^{\beta_1}_{\mathcal{X}^{\beta_1} X}(y_2) \in W^{\beta_1}_\alpha(V)$. This contradicts $W^{\beta_1}_\alpha(U) \cap W^{\beta_1}_\alpha(V) = \emptyset$.

We have a new category, $\text{Preord}^\alpha$. It is interesting to see that the order $\sqsubseteq$ on an element of $\text{Preord}^\alpha$ *respects* in a certain sense the first preorder.
For every ordinals $\beta < \alpha$, $\gamma$ such that $\beta + \gamma = \alpha$, we have a natural transformation $\{ \cdot \}^\beta_\gamma : K^\beta \to K^\alpha$. We can interpret $\gamma$ as how far away $\mathcal{K}^\alpha X$ is from $\mathcal{K}^\beta X$. It is interesting to see in a proof by induction on an ordinal, how difficult it is for the case successor and the case limit. Sometimes the limit case is less difficult than the successor case. There are probably many further perspectives, with the functor $K^\alpha$.

A  Appendix

A.1  Topology on the set of compact subsets

In [Mer10], one defines the following. If $(X, \tau)$ is a topological space, one equips the set $\mathcal{X}X$ of non-empty compact subsets of $(X, \tau)$ with a topology $\tau^1$. A sub-basis for this topology is given by the family of sets $W(U) = \{ A \in \mathcal{X}X \mid A \cap U \neq \emptyset \}$ for $U \in \tau$. It is proved that if $(X, \tau)$ is $T_2$ then $(\mathcal{X}X, \tau^1)$ is also $T_2$ (Lemme 4.12), if $(X, \tau)$ is compact then $(\mathcal{X}X, \tau^1)$ is also compact (Théorème 4.23) and if $(X, \tau)$ is discrete then $(\mathcal{X}X, \tau^1)$ is also discrete. We want to add the following lemma. By Tychonoff’s theorem, we know that the Cartesian product of compact spaces is compact. In fact we have the following.

**Lemma A.1**

Let $(X, \tau)$ and $(X', \tau')$ be two topological spaces. The map

$$\kappa : \mathcal{X}X \times \mathcal{X}X' \longrightarrow \mathcal{X}(X \times X')$$

$$(A, B) \longmapsto A \times B$$

is continuous.

**Proof.** It is sufficient to check that the preimage of a sub-basis open set is open. The open subsets of $X \times X'$ have the form $\bigcup_{i \in I} U_i \times U'_i$ with $I \neq \emptyset$, $U_i \in \tau$ and $U'_i \in \tau'$. Let us check that the preimage of $W(\bigcup_{i \in I} U_i \times U'_i)$ and the preimage of $W(X \times X', \bigcup_{i \in I} U_i \times U'_i)$ are open.

- We have $\kappa^{-1}(W(X \times X', \bigcup_{i \in I} U_i \times U'_i)) = \{(A, B) \in \mathcal{X}X \times \mathcal{X}X' \mid A \times B \cap (\bigcup_{i \in I} U_i \times U'_i) \neq \emptyset \}$. We claim
  $$\kappa^{-1}(W(X \times X', \bigcup_{i \in I} U_i \times U'_i)) = \bigcup_{i \in I} W(X, U_i) \times W(X', U'_i).$$
  
  ∴ Let $(A, B) \in \kappa^{-1}(W(X \times X', \bigcup_{i \in I} U_i \times U'_i))$. We have
  $$\exists (a, b) \in A \times B, (a, b) \in \bigcup_{i \in I} U_i \times U'_i \implies \exists (a, b) \in A \times B, \exists i \in I \text{ s.t. } (a, b) \in U_i \times U'_i$$
  $$\implies \exists i \in I \text{ s.t. } A \cap U_i \neq \emptyset, B \cap U'_i \neq \emptyset.$$
  So $(A, B) \in \bigcup_{i \in I} W(X, U_i) \times W(X', U'_i)$.

- We have $\kappa^{-1}(W(\bigcup_{i \in I} U_i \times U'_i)) = \{(A, B) \in \mathcal{X}X \times \mathcal{X}X' \mid A \times B \cap \bigcup_{i \in I} U_i \times U'_i \neq \emptyset \}$. We claim
  $$\kappa^{-1}(W(\bigcup_{i \in I} U_i \times U'_i)) = \bigcup_{|I'| < \infty} \kappa^{-1}(W(\bigcup_{i \in I'} U_i \times U'_i)).$$

The inclusion from right to left is clear. The other inclusion comes from the fact that an element $(A, B)$ in $\kappa^{-1}(W(\bigcup_{i \in I} U_i \times U'_i))$ is compact so admits a finite sub-covering of $\{U_i \times U'_i\}_{i \in I}$. So we just need to check that $\kappa^{-1}(W(\bigcup_{i \in I} U_i \times U'_i))$ is open for $I$ finite.

Let us rename $I' = J$ with $|J| < \infty$. We claim

$$\kappa^{-1}(W(\bigcup_{j \in J} U_j \times U'_j)) = \bigcup_{\emptyset \neq L \subseteq J} W(\bigcup_{\emptyset \neq L \subseteq J} U_L) \times W(\bigcup_{\emptyset \neq L \subseteq J} U'_L).$$
Corollary A.2
Let $(M, \cdot, \tau)$ be a topological monoid. For $A, B \in \mathcal{M}$ we define

$$A \cdot B := \{a \cdot b \mid (a, b) \in A \times B\}.$$ 

This induces a continuous map: $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$.

Proof. The fact that $A \cdot B$ is indeed in $\mathcal{M}$ follows from the fact that it is the direct image of a compact by a continuous application.

Our map can be written as the composition of the following maps:

$$\mathcal{M} \times \mathcal{M} \xrightarrow{n_{A,B}} \mathcal{M} \xrightarrow{K} \mathcal{M}$$

$$(A, B) \xrightarrow{n_{A,B}} A \times B \xrightarrow{K} A \cdot B.$$ 

The first is continuous by Lemma A.1 and the second is continuous by definition of topological monoid and the fact that $K$ is a functor. So our map is continuous as the composition of continuous maps.

On the category Top we have the monad $\mathbb{K} = (K, \{\cdot\}, U)$. For a topological space $(X, \tau)$, $K(X, \tau) := (\mathcal{X}, \tau^\dagger)$. The functor $K$ restricts and co-restrict to the category $\text{CompHaus}$, one can restrict the monad $\mathbb{K}$ to $\text{CompHaus}$ in this case the $\mathbb{K}$-algebras are the compact and $T_2$ topological spaces equipped with an almost complete sup-semilattice structure for which the suprema map $(\mathcal{X}, \tau^\dagger) \to (X, \tau)$ is continuous. Given a $\mathbb{K}$-algebra $(X, a)$ on the category $\text{CompHaus}$, the suprema of a non-empty subset $X' \subset X$ is given by $a(X')$, where $\overline{X'}$ is the closure of $X'$.

A.2 Ordinals

One can find a good description of ordinals in [Jec03] chapter 2. We will use the corresponding notations. In particular, we will write $\omega$ for the smallest infinite ordinal. This means $\omega = \bigcup_{n \in \mathbb{N}} n$. We write $\beta \leq \alpha$ for saying $\beta \in \alpha$ and we write $\beta < \alpha$ for saying $\beta \in \alpha$ or $\beta = \alpha$. In general, when we define or prove something for ordinals, it is by induction, considering the cases $0$, successor and limits. The ordinals are ordered by the relation $\in$. We recall that a non-empty class of ordinals has always an $\in$-minimal element. We recall a proposition that will be used in section 4.

Proposition A.3
Let $\alpha, \beta$ be two ordinals such that $\beta < \alpha$. There exists a unique ordinal $\gamma$ such that $\beta + \gamma = \alpha$.

Proof. See Lemma 2.25. (ii) in [Jec03].
List of categories

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List of monads

\( P = (\mathcal{P}, \{\cdot\}, \cup) \) on Set: \( \mathcal{P} \) is the functor assigning to a set, its set of subsets, and assigning to a function, the function sending subset to direct image, i.e. if \( f : X \to X' \) is function and \( A \subset X \), then \( \mathcal{P} f (A) = \{ f(a) \mid a \in A \} \). For a set \( X \), \( \{\cdot\}_X \) sends \( x \) to \( \{x\} \) and \( \cup_X \) sends \( \mathcal{A} \) to \( \cup \mathcal{A} \).

\( P₀ = (\mathcal{P}_0, \{\cdot\}, \cup) \) on Set: \( \mathcal{P}_0 \) is the functor assigning to a set, its set of non-empty subsets. It is defined as \( \mathcal{P} \) on morphisms.

\( F = (\mathcal{F}, <, >, \nu) \) on Set: \( \mathcal{F} X \) gives the set of filters on \( X \) (i.e on \( \mathcal{P} X \) with inclusion). If \( f : X \to X' \) is a function and \( \xi \in \mathcal{F} X \), \( \mathcal{F} f (\xi) = \{ A' \subset X' \mid f^{-1}(A') \in \xi \} \). For \( x \in X \), \( <, >_X (x) \) is the principal filter generated by \( x \). For \( \Xi \in \mathcal{F F} X \), \( \nu_X (\Xi) = \{ A' \subset X' \mid \{ \xi \in \mathcal{F} X \mid A' \in \xi \} \in \Xi \} \).

\( F₀ = (\mathcal{F}_0, <, >, \nu) \) on Set: \( \mathcal{F}_0 X \) gives the set of proper filters on \( X \), i.e. all the filters except the power set of \( X \). It defined as \( \mathcal{F} \) on morphisms.

\( U = (\mathcal{U}, <, >, \nu) \) on Set: \( \mathcal{U} X \) gives the set of ultra-filters on \( X \). It defined as \( \mathcal{F} \) on morphisms.

\( K = (K, \{\cdot\}, \cup) \) on Top: \( K \) is a functor assigning to a topological space \((X, \tau)\) a topological space \((X, \tau^1)\) with underlying set \( X \) the set of non-empty compact subsets. This functor is defined in Appendix A.1. On continuous maps it is defined on the underlying sets as \( \mathcal{P} \). For a topological space \( X \), the maps \( \{\cdot\}_X \) and \( \cup_X \) are defined on the underlying sets as in the monad \( \mathcal{P} \).

The monad \( K \) restricts and corestricts to the category \( \text{CompHaus} \).

\( S = (S, \delta, \nu) \) on Preord: \( S \) is a functor assigning to preordered set \( X \), a preordered set having as underlying set, the set of subset \( X' \) having the following property. If there exists a set \( I \subset X \) such that \( X' \subset \bigcup_{x \in I} \uparrow x \), then there exists a finite subset \( I' \subset I \) such that \( X' \subset \bigcup_{x \in I'} \uparrow x \). This functor is defined in paragraph 3.1. On increasing maps, it is defined on the underlying sets as \( \mathcal{P} \). For a preordered set \( X \), the maps \( \delta_X \) and \( \nu_X \) are defined on the underlying set as the map \( \{\cdot\}_X \) and the map \( \cup_X \) in the monad \( \mathcal{P} \).
References


