Order-adjoint monads and injective objects

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Abstract

Given a monad $T$ on $\text{Set}$ whose functor factors through the category of ordered sets with left adjoint maps, the category of Kleisli monoids is defined as the category of monoids in the hom-sets of the Kleisli category of $T$. The Eilenberg-Moore category of $T$ is shown to be strictly monadic over the category of Kleisli monoids. If the Kleisli category of $T$ moreover forms an order-enriched category, then the monad induced by the new situation is Kock-Zöberlein. Injective objects in the category of Kleisli monoids with respect to the class of initial morphisms then characterize the objects of the Eilenberg-Moore category of $T$, a fact that allows us to recuperate a number of known results, and present some new ones.

Keywords: Galois adjunction; Kock-Zöberlein monad; monoid in a category; fibration

1 Introduction

In [6], Gähler introduced the concept of a “monadic topology” based on the observation that the category $\text{Top}$ of topological spaces and continuous maps could be entirely defined in terms of the filter monad $F$ and its Kleisli category $\text{Set}^F$. The Eilenberg-Moore algebras of $F$ also have a topological facet, as continuous lattices play a central role in the study of ordered topological spaces (see in particular [7] and [21]). A similar situation occurs for the powerset monad $P$, whose “monadic topologies” are preordered sets, and Eilenberg-Moore algebras are complete lattices— that is, particular preordered sets. This last example shows that “monadic topologies” do not necessarily have an obvious topological nature, so we refer to them as Kleisli monoids instead, as they are monoids in the hom-sets of a Kleisli category (see 3.1 for details).

The previous monads defined on $\text{Set}$ also have a counterpart on their category of Kleisli monoids: the filter monad on $\text{Set}$ and the filter of open sets on $\text{Top}$, the powerset monad on $\text{Set}$ and the down-set monad on $\text{PrOrd}$. Indeed, these corresponding monads have the same category of Eilenberg-Moore algebras: the category $\text{Cnt}$ of continuous lattices and continuous sup-maps for the filter and filter of open set monads, and the category $\text{Sup}$ of complete lattices and sup-maps for the powerset and down-set monads. In both cases the “structured” version of the $\text{Set}$-based monad possesses a very desirable property: it is of Kock-Zöberlein type.

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In the context of Kock-Zöberlein monads on order-enriched categories, there moreover is a one-to-one correspondence between certain injective objects in the category, and the Eilenberg-Moore algebras of the monad (see [21] and [5]). For instance, continuous lattices are precisely the injective objects of \( \text{Top}_0 \), and complete lattices are those of \( \text{Ord} \). Kleisli monoids appear here as \textit{separated} topological spaces, and \textit{separated} preordered sets, respectively.

The purpose of this work is to present a general setting in which the aforementioned interactions between Kleisli monoids, Eilenberg-Moore algebras, and injective objects can be studied (the results we obtain for the latter are complementary to those of the cited references). Briefly put, we describe the passage from an \textit{order-adjoint} monad \( T \) on \( \text{Set} \) to a Kock-Zöberlein monad \( T' \) on the category \( \text{Set}(T) \) of Kleisli monoids, and then characterize the Eilenberg-Moore category \( \text{Set}^T \) in terms of injective objects in \( \text{Set}(T) \). The article is structured as follows, with the main results presented in the form of three Theorems:

- order-adjoint monads \( T \) and their algebras (Section 2)
- \( \text{Set}^T \) is monadic over \( \text{Set}(T) \) (Theorem 4.5)
- \( \text{Set}^T \) describes injective objects in \( \text{Set}(T) \) (Theorem 5.3).

The examples that appear in the first sections are meant to support theoretical aspects, and are often exploited further on in the text without necessarily explicit mention. The more concrete examples illustrating the theorems are gathered at the end of the article, in Section 6. Therein, we selected a number of known and new results to present in the light of Theorems 4.5, 4.9 and 5.3.

### 2 Order-adjoint monads

#### 2.1 Monads. A monad \( T \) on a category \( X \) is a triple \((T, \eta, \mu)\), with \( T : X \rightarrow X \) a functor, while the \textit{unit} \( \eta : 1_X \rightarrow T \) and \textit{multiplication} \( \mu : TT \rightarrow T \) of \( T \) are natural transformations satisfying

\[
\mu \cdot T \eta = 1_T = \mu \cdot \eta T \quad \text{and} \quad \mu \cdot T \mu = \mu \cdot \mu T .
\]

A \textit{monad morphism} \((R, \sigma) : S \rightarrow T\) from a monad \( S = (S, \delta, \nu) \) on \( A \) to a monad \( T = (T, \eta, \mu) \) on \( X \) is given by a functor \( R : X \rightarrow A \) together with a natural transformation \( \sigma : SR \rightarrow RT \) such that

\[
R \eta = \sigma \cdot \delta R \quad \text{and} \quad R \mu \cdot \sigma T \cdot S \sigma = \sigma \cdot \nu R .
\]

In the case where \( A = X \) and \( R \) is the identity, one writes \( \sigma : S \rightarrow T \) rather than \((1_X, \sigma) : S \rightarrow T\). A monad can also be described by way of a \textit{Kleisli triple} \((T, \eta, (-)^T)\) on \( X \) (Exercise 1.3.12 in [16]), that is,

1. a function \( T : \text{ob} \ X \rightarrow \text{ob} \ X \),
2. for each \( X \)-object \( X \), an \( X \)-morphism \( \eta_X : X \rightarrow TX \),
(iii) an extensions operation \((-)^T\) that sends an \(X\)-morphism \(f : X \rightarrow TY\) to an \(X\)-morphism \(f^T : TX \rightarrow TY\), subject to the conditions
\[
(\eta_X)^T = 1_{TX}, \quad f^T \cdot \eta_X = f \quad \text{and} \quad g^T \cdot f^T = (g^T \cdot f)^T
\]
for all \(f : X \rightarrow TY, g : Y \rightarrow TZ\). Every Kleisli triple \((T, \eta, (-)^T)\) yields a monad \(T = (T, \eta, \mu)\) via
\[
Tf := (\eta_Y \cdot f)^T \quad \text{and} \quad \mu_X := (1_{TX})^T,
\]
and every monad \(T = (T, \eta, \mu)\) defines a Kleisli triple thanks to
\[
f^T := \mu_Y \cdot Tf.
\]
These processes are inverse of one another, and from now on we freely switch between the two descriptions.

In the case where two Kleisli triples \((S, \delta, (-)^S)\) and \((T, \eta, (-)^T)\) are defined on the same category \(X\), a family \((\sigma_X : SX \rightarrow TX)_{X \in \text{ob}X}\) defines a monad morphism \(\sigma : S \rightarrow T\) if and only if the equalities
\[
\eta_X = \sigma_X \cdot \delta_X \quad \text{and} \quad (\sigma_Y \cdot f)^T \cdot \sigma_X = \sigma_Y \cdot f^S
\]
hold for all \(X\)-objects \(X\) and \(X\)-morphisms \(f : X \rightarrow SY\).

### 2.2 Eilenberg-Moore and Kleisli categories.

Given a monad \(T = (T, \eta, \mu)\) on a category \(X\), an Eilenberg-Moore algebra (or a \(T\)-algebra) is a pair \((X, a)\), with \(X\) an object of \(X\), and \(a : TX \rightarrow X\) a structure morphism that satisfies
\[
1_X = a \cdot \eta_X \quad \text{and} \quad a \cdot Ta = a \cdot \mu_X.
\]
In particular, the pair \((TX, \mu_X)\) forms an Eilenberg-Moore algebra (called the free \(T\)-algebra on \(X\)). A morphism of Eilenberg-Moore algebras \(f : (X, a) \rightarrow (Y, b)\) is an \(X\)-morphism \(f : X \rightarrow Y\) such that
\[
f \cdot a = b \cdot Tf.
\]
The category of Eilenberg-Moore algebras and their morphisms is denoted by \(X^T\) and is called the Eilenberg-Moore category of \(T\). In the case the monad \(T\) is given by a Kleisli triple, the conditions for an \(X\)-morphism \(a : TX \rightarrow X\) to form an Eilenberg-Moore structure can be expressed as
\[
1_X = a \cdot \eta_X \quad \text{and} \quad \forall f, g \in \text{X}(Y, TX) \quad (a \cdot f = a \cdot g \implies a \cdot f^T = a \cdot g^T).
\]
Given a monad \(S = (S, \delta, \nu)\) on \(A\) and a monad \(T\) on \(X\), a functor \(\overline{R} : X^T \rightarrow A^S\) is algebraic over a functor \(R : X \rightarrow A\) if it makes the diagram
\[
\begin{array}{ccc}
X^T & \xrightarrow{\overline{R}} & A^S \\
\downarrow & & \downarrow \\
X & \xrightarrow{R} & A
\end{array}
\]
commute (the vertical arrows represent the respective forgetful functors). Any monad morphism $(R, \sigma) : S \rightarrow T$ from a monad $S$ on $A$ to a monad $T$ on $X$ induces such an algebraic functor; this is defined on objects by

$$\overline{R}(X, a) = (RX, Ra \cdot \sigma_X),$$

and necessarily sends an $X$-morphism $f$ to $Rf$. Conversely, every functor $\overline{R} : X^T \rightarrow A^S$ that is algebraic over $R : X \rightarrow A$ is induced by a monad morphism $(R, \sigma)$: if $\mu_X : SRTX \rightarrow RTX$ denotes the $A$-morphism given by $\overline{R}(TX, \mu_X) = (RTX, \overline{\mu}_X)$, then one can define the components of $\sigma : SR \rightarrow RT$ by

$$\sigma_X := \overline{\mu}_X \cdot SR\eta_X.$$

The objects of the Kleisli category $X_T$ associated to the monad $T$ are the objects of $X$, and morphisms $f : X \rightarrow Y$ in $X_T$ are those $X$-morphisms $f : X \rightarrow TY$. Kleisli composition of $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in $X_T$ is defined via the composition in $X$ as

$$g \circ f := \mu_Z \cdot Tg \cdot f = g^T \cdot f.$$

The identity $1_X : X \rightarrow X$ in this category is just the component $\eta_X : X \rightarrow TX$ of the unit.

### 2.3 Kock-Zöberlein monads.

A category $A$ is a preorder-enriched category if its hom-sets carry a preorder that is preserved by composition on each side: for $f, f' : X \rightarrow Y$, $h : Y \rightarrow Z$ and $g : W \rightarrow X$, one must have

$$f \leq f' \implies h \cdot f \cdot g \leq h \cdot f' \cdot g.$$

An adjoint situation $f \dashv g : Y \rightarrow X$ in $A$ is a pair of $A$-morphisms $f : X \rightarrow Y$, the left adjoint, and $g : Y \rightarrow X$, the right adjoint, satisfying the inequalities

$$1_X \leq g \cdot f \quad \text{and} \quad f \cdot g \leq 1_Y.$$

In a preorder-enriched category, adjoints are usually only determined up to equivalence, while they are uniquely determined in an order-enriched category (that is, a preorder-enriched category in which the preorder on the hom-sets is antisymmetric). A functor $S : A \rightarrow A$ is a 2-functor if it preserves the preorder on hom-sets:

$$f \leq g \implies Sf \leq Sg$$

for all $f, g \in A(X, Y)$. Such a functor then also preserves adjoint situations.

A monad $S = (S, \delta, \nu)$ on a preorder-enriched category $A$ is Kock-Zöberlein if $S$ is a 2-functor, and for every $A$-object $X$ there is a chain of adjoint situations:

$$S\delta_X \dashv \nu_X \dashv \delta_{SX}.$$

The previous chain condition can be replaced by any of the following equivalent expressions (see [13] or Lemma 4.1.1 of [5]):

$$\forall X \in \text{ob} \ A \ (S\delta_X \leq \delta_{SX}) \iff \forall X \in \text{ob} \ A \ (S\delta_X \dashv \nu_X) \iff \forall X \in \text{ob} \ A \ (\nu_X \dashv \delta_{SX}).$$

In the case where $S$ is a Kock-Zöberlein monad on an order-enriched category $A$, the Eilenberg-Moore algebras are exactly those pairs $(X, a : SX \rightarrow X)$ for which $a$ is left adjoint to the split mono $\delta_X$:

$$1_{SX} \leq \delta_X \cdot a \quad \text{and} \quad a \cdot \delta_X = 1_X.$$
2.4 Order-adjoint monads. Let $\text{Ord}$ denote the category of ordered sets with monotone maps, and $\text{Ord}_*$ the subcategory of $\text{Ord}$ with same objects but whose maps are left adjoint. Explicitly a map $f : X \to Y$ is a morphism of $\text{Ord}_*$ if it is monotone and there exists a monotone map, denoted by $f^* : Y \to X$, satisfying

$$1_X \leq f^* \cdot f \quad \text{and} \quad f \cdot f^* \leq 1_Y.$$  

Alternatively, one asks that $f : X \to Y$ and $f^* : Y \to X$ are just maps such that $f(x) \leq y \iff x \leq f^*(y)$ for all $x \in X$, $y \in Y$ (monotonicity of $f$ and $f^*$ being a consequence of the equivalence).

A functor $T : \text{Set} \to \text{Set}$ that factors through $\text{Ord}_*$ does so via a functor $S : \text{Set} \to \text{Ord}_*$ making the diagram commute (where $|−|$ denotes the forgetful functor). For convenience, such a functor $T$ is understood to be given with a fixed $S$, which is moreover identified with $T$; for example, we talk about “the right adjoint $(Tf)^*$ of $Tf : TX \to TY$” to mean “the image via the forgetful functor of the right adjoint $(Sf)^*$ of $Sf : SX \to SY$”. The hom-sets $\text{Set}(X, TY)$ are then equipped with the pointwise order, so that for $f, f' \in \text{Set}(X, TY)$, one has

$$f \leq f' \iff \forall x \in X \ (f(x) \leq f'(x)).$$

A monad $T = (T, \eta, \mu)$ on $\text{Set}$ is order-adjoint if $T$ factors through $\text{Ord}_*$ and each component $\mu_X$ of the monad multiplication is a monotone map with right adjoint $\mu^*_X$. Such a monad is enhanced if moreover the extension operation $(−)^T$ preserves the order on the hom-sets $\text{Set}(X, TY)$:

$$f \leq g \implies f^T \leq g^T$$

for all maps $f, g : X \to TY$. In this case, $\text{Set}_T$ becomes an order-enriched category. An order-adjoint monads is not necessarily enhanced (the double-dualization monad provides such an example, see 6.5), but even if it is, its functor needs not preserve adjoint situations, so that $T(Tf)^* = (TTf)^*$ does not hold in general.

2.5 Remark. Order-adjoint monads are similar in spirit to Kock-Zöberlein monads on $\text{Ord}$. We will see in Section 4 how this parallel can be formalized.

2.6 Lemma. An $\text{Ord}_*$-morphism $f : X \to Y$ is a retraction in $\text{Set}$ if and only if $f \cdot f^* = 1_Y$.

Proof. On one hand, if $f \cdot f^* = 1_Y$, then $f$ is a retraction by definition. On the other hand, if there is a map $g : Y \to X$ with $f \cdot g = 1_Y$, then $g \leq f^*$. Composing the latter inequality with $f$ on the left, we get $1_Y \leq f \cdot f^*$; since $f \cdot f^* \leq 1_Y$ by definition of the right adjoint, we can conclude that $f \cdot f^* = 1_Y$. \qed
2.7 Lemma. If $T$ is an order-adjoint monad and $a : TX \to X$ an Eilenberg-Moore algebra structure, then for the map $a^\circ : X \to TX$ defined by

$$a^\circ := \mu_X \cdot (Ta)^* \cdot \eta_X ,$$

one has

$$1_{TX} \leq a^\circ \cdot a \quad \text{and} \quad a \cdot a^\circ = 1_X .$$

Proof. On one hand, $Ta \cdot T\eta_X = 1_{TX}$ implies $Ta \cdot (Ta)^* = 1_{TX}$ by Lemma 2.6, and

$$a \cdot a^\circ = a \cdot \mu_X \cdot (Ta)^* \cdot \eta_X = a \cdot Ta \cdot (Ta)^* \cdot \eta_X = a \cdot \eta_X = 1_X .$$

On the other hand, we observe that

$$1_{TX} = \mu_X \cdot \eta_X \leq \mu_X \cdot (Ta)^* \cdot Ta \cdot \eta_X = \mu_X \cdot (Ta)^* \cdot \eta_X \cdot a = a^\circ \cdot a .$$

2.8 Remark. As emphasized in the hypothesis of the previous lemma, $a^\circ$ is defined as a set-map. The statement of the lemma suggests that this map is a right adjoint for $a$, but for this to be true, one has to first verify that $X$ can be equipped with an order making both $a$ and $a^\circ$ monotone. This is the subject of the following results.

2.9 Proposition. If $T$ is an order-adjoint monad, then one has for any Eilenberg-Moore algebra structure $a : TX \to X$ the equivalence

$$\eta_X(y) \leq a^\circ(x) \iff a^\circ(y) \leq a^\circ(x) .$$

This provides $X$ with the order defined for all $x,y \in X$ by

$$y \leq x \iff a^\circ(y) \leq a^\circ(x) . \quad (*)$$

The corresponding order on $TX$ induced by $\mu_X : TTX \to TX$ is the original order of $TX$ (that is, the order determined by the functor $S$).

Proof. Let us prove the first equivalence. Suppose that $\eta_X(y) \leq a^\circ(x)$ holds; we recall the inequality $1_{TX} \leq a^\circ \cdot a$ of Lemma 2.7, and can write, after composing with $\mu_X \cdot (Ta)^*$:

$$a^\circ(y) = \mu_X \cdot (Ta)^* \cdot \eta_X(y) \leq \mu_X \cdot (Ta)^* \cdot a^\circ(x) \leq a^\circ \cdot a \cdot \mu_X \cdot (Ta)^* \cdot a^\circ(x) = a^\circ \cdot a \cdot a^\circ(x) = a^\circ(x) ,$$

(by using that $a \cdot \mu_X = a \cdot Ta$, together with $Ta \cdot (Ta)^* = 1_{TX}$ by Lemma 2.6, and $a \cdot a^\circ = 1_X$ by Lemma 2.7). Suppose now that $a^\circ(y) \leq a^\circ(x)$; since $Ta \cdot T\eta_X = 1_{TX}$ implies $T\eta_X \leq (Ta)^*$, we have

$$\eta_X(y) = \mu_X \cdot T\eta_X \cdot \eta_X(y) \leq \mu_X \cdot (Ta)^* \cdot \eta_X(y) = a^\circ(y) .$$

The definition given in $(*)$ provides $X$ with a preorder (the initial preorder induced by $a^\circ : X \to TX$, see also 3.1 and 3.8). Antisymmetry then follows from the fact that $a^\circ$ is an embedding: $a^\circ(x) = a^\circ(y)$ implies $x = y$ by composing with $a$ on the left.
To verify the last claim, recall that by hypothesis $\mu_X$ has a right adjoint $\mu_X^*$. On one hand, naturality of the unit yields $T\mu_X \cdot \eta_{TX} \cdot \mu_X^* = \eta_{TX}$ (as $\mu_X \cdot \eta_{TX} = 1_{TX}$ implies $\mu_X \cdot \mu_X^* = 1_{TX}$ by Lemma 2.6) so that $\eta_{TX} \cdot \mu_X^* \leq (T\mu_X)^* \cdot \eta_{TX}$; on the other hand, the multiplication law of the monad gives us the inequality $\mu_X \cdot (T\mu_X)^* \leq \mu_X^* \cdot \mu_X$; thus, one has

$$\mu_X^* = \mu_X \cdot \eta_{TX} \cdot \mu_X^* \leq \mu_X \cdot (T\mu_X)^* \cdot \eta_{TX} \leq \mu_X^* \cdot \mu_X \cdot \eta_{TX} = \mu_X^*,$$

and one can therefore conclude that $\mu_X^* = \mu_X^*$ by definition of $\mu_X^*$. Since $\mu_X \cdot \mu_X^* = 1_{TX}$, one has for $\xi, \eta \in TX$ that

$$\eta \leq \xi \iff \mu_X^*(\eta) \leq \mu_X^*(\xi).$$

As these inequalities use the original order on $TX$, we are done. \hfill \square

2.10 Proposition. A monad $T$ on $\text{Set}$ is order-adjoint if and only if the forgetful functor $R : \text{Set}^T \to \text{Set}$ factors through $\text{Ord}^*$.

Proof. If $R$ factors through $\text{Ord}^*$, the free algebra morphisms $Tf : (TX, \mu_X) \to (TY, \mu_Y)$ and $\mu_X : (TTX, \mu_{TX}) \to (TX, \mu_X)$ are $\text{Ord}^*$-morphisms, a fact that yields the required conditions for $T$ to be order-adjoint.

Consider now an order-adjoint monad $T$. To see that a structure $a : TX \to X$ of an Eilenberg-Moore algebra is an $\text{Ord}^*$-morphism, we first verify that $a$ and $a^\circ$ are monotone with respect to the order on $X$ defined in Proposition 2.9; Lemma 2.7 then allows us to conclude that one has $a \dashv a^\circ$. Thus, consider $\xi, \eta \in TX$ such that $\xi \leq \eta$; one observes that

$$\eta_X \cdot a(\xi) = Ta \cdot \eta_{TX}(\xi) \quad \text{(naturality of $\eta$)}$$
$$\leq Ta \cdot \mu_X^*(\xi) \quad \text{($\eta_{TX} \leq \mu_X^*$)}$$
$$\leq Ta \cdot \mu_X^*(\eta) \quad \text{($\xi \leq \eta$)}$$
$$\leq a^\circ \cdot a \cdot Ta \cdot \mu_X^*(\eta) \quad \text{($1_{TX} \leq a^\circ \cdot a$)}$$
$$= a^\circ \cdot a \cdot \mu_X \cdot \mu_X^*(\eta) \quad \text{($a \cdot Ta = a \cdot \mu_X$)}$$
$$= a^\circ \cdot a(\eta) \quad \text{($\mu_X \cdot \mu_X^* = 1_{TX}$)}.$$

By Proposition 2.9, this means precisely that $a(\xi) \leq a(\eta)$. The definition of the order on $X$ immediately yields that $a^\circ$ is monotone, so $a^\circ$ is indeed the right adjoint $a^*$ of $a$.

To verify that morphisms of Eilenberg-Moore algebras are $\text{Ord}^*$-morphisms, consider a map $f : X \to Y$ such that $f \cdot a = b \cdot Tf$. From this equality, one deduces that

$$f = b \cdot Tf \cdot a^*,$$

so that $f$, being a composite of monotone maps, is monotone. Let us now check that the monotone map

$$f^\circ := a \cdot (Tf)^* \cdot b^*$$

is right adjoint to $f$. For this, note that

$$Tf \cdot a^* \leq b^* \cdot f \quad \text{(*)}$$
by adjunction of the $T$-algebra morphism condition. One therefore has

$$1_X \leq a \cdot (Tf)^* \cdot Tf \cdot a^* \leq a \cdot (Tf)^* \cdot b^* \cdot f = f^* \cdot f$$

as well as

$$f \cdot f^* = f \cdot a \cdot (Tf)^* \cdot b^* = b \cdot Tf \cdot (Tf)^* \cdot b^* \leq 1_Y ,$$

which proves the claim.

\[\square\]

3 Kleisli monoids

3.1 The category of Kleisli monoids. Given an order-adjoint monad $T$ on $\text{Set}$, a Kleisli monoid (or simply a $T$-monoid) is a pair $(X, \alpha)$ made up of a set $X$ and a structure map $\alpha : X \to TX$ that satisfies

$$\eta_X \leq \alpha \quad \text{and} \quad \alpha^T \cdot \alpha \leq \alpha .$$

In our context, it is crucial to remark that in the presence of the first condition the second may be expressed as an equality: $\alpha^T \cdot \alpha = \alpha$. A Kleisli morphism $f : (X, \alpha) \to (Y, \beta)$ is a map $f : X \to Y$ with

$$Tf \cdot \alpha \leq \beta \cdot f ,$$

and $f$ composes with another Kleisli morphism $g : (Y, \beta) \to (Z, \gamma)$ as in $\text{Set}$. The category of Kleisli monoids and their morphisms is denoted by $\text{Set}(T)$.

The underlying set of a Kleisli monoid $(X, \alpha)$ can be equipped with the initial preorder induced by $\alpha : X \to TX$: for $x, y \in X$

$$x \leq y \iff \alpha(x) \leq \alpha(y) .$$

This preorder becomes an order exactly when $\alpha : X \to TX$ is a monomorphism; in this case, the Kleisli monoid $(X, \alpha)$ is said to be separated. The full subcategory of $\text{Set}(T)$ whose objects are the separated Kleisli monoids is denoted by $\text{Set}(T)_0$.

3.2 A word on terminology. As mentioned in the Introduction, the category $\text{Set}(T)$ finds its origin in [6] (albeit with different hypotheses on the monad $T$), wherein Gähler introduces the category to define certain generalizations of topological spaces, and the objects of the category are coined “monadic topologies”. The category later appears in [22] (again, with slightly different hypotheses on $T$), in which, to emphasize the categorical approach, the name “Kleisli algebra” was used. This terminology was the one initially chosen for this article and [20], but the simultaneous occurrence of “Eilenberg-Moore” and “Kleisli” algebras turned out to be unwieldy, and added unwanted confusion to the term “$T$-algebra”. Hence, the objects of $\text{Set}(T)$ are now called Kleisli monoids, since they are precisely the monoids in the ordered hom-sets $\text{Set}_T(X, X)$ considered as categories.

3.3 Examples.

(1) Since one obviously has $\eta_X \leq \eta_X$ and $\eta^*_X \cdot \eta_X \leq \eta_X$, the pair $(X, \eta_X)$ forms a Kleisli monoid, called the discrete Kleisli monoid.
(2) By naturality of \( \eta \), any map \( f : X \to Y \) yields a Kleisli morphism \( f : (X, \eta_X) \to (Y, \eta_Y) \). Given a Kleisli monoid \( (X, \alpha) \), the identity \( 1_X : X \to X \) is also a Kleisli morphism
\[
1_X : (X, \eta_X) \to (X, \alpha)
\]
since \( T1_X \cdot \eta_X = \eta_X \leq \alpha = \alpha \cdot 1_X \). Similarly, the structure morphism \( \alpha \) is a morphism of Kleisli monoids
\[
\alpha : (X, \alpha) \to (TX, \mu_X)
\]
because \( T\alpha \cdot \alpha \leq \mu_X \cdot \alpha \) is equivalent to \( \alpha^T \cdot \alpha \leq \alpha \) by adjunction.

(3) The previous considerations on \( \eta \) imply the existence of a functor \( D : \text{Set} \to \text{Set}(\mathbb{T}) \) that sends a set \( X \) to the discrete Kleisli monoid \( (X, \eta_X) \), and remains identical on maps. This functor is left adjoint to the forgetful functor \( \text{Set}(\mathbb{T}) \to \text{Set} \) (that sends \( (X, \alpha) \) to its underlying set \( X \), and leaves maps unchanged).

(4) Given a \( \mathbb{T} \)-algebra \( (X, a) \), the pair \( (X, a^*) \) defines a Kleisli monoid (Proposition 2.10 guarantees that \( a \) does indeed have a right adjoint \( a^* \)): the Eilenberg-Moore conditions imply \( a \cdot \eta_X \leq 1_X \) and \( a \cdot \mu_X \leq a \cdot Ta \), so that
\[
\eta_X \leq a^* \quad \text{and} \quad \mu_X \cdot T(a^*) \cdot a^* \leq (a^* \cdot a \cdot Ta) \cdot T(a^*) \cdot a^* = a^*.
\]
Similarly, a morphism \( f : (X, a) \to (Y, b) \) of Eilenberg-Moore algebras yields a morphism \( f : (X, a^*) \to (Y, b^*) \), as the condition \( b \cdot Tf \leq f \cdot a \) is equivalent to
\[
Tf \cdot a^* \leq b^* \cdot f.
\]
The right adjoint operation on structures therefore defines a factorization of the forgetful functor \( R : \text{Set}^\mathbb{T} \to \text{Set} \) through \( Q : \text{Set}^\mathbb{T} \to \text{Set}(\mathbb{T}) \). In fact, since \( a \cdot a^* = 1_X \) (Lemma 2.6), the structure \( a^* \) is a monomorphism, so the forgetful functor also factors through the category of separated Kleisli monoids, that is, \( Q \) can be seen as a functor \( Q : \text{Set}^\mathbb{T} \to \text{Set}(\mathbb{T})_0 \).

3.4 Lemma. For a Kleisli monoid \( (X, \alpha) \), one has
\[
\eta_X(x) \leq \alpha(y) \iff \alpha(x) \leq \alpha(y)
\]
for all \( x, y \in X \).

Proof. Since \( \eta_X \leq \alpha \), one immediately observes that for \( x, y \in X \), if \( \alpha(x) \leq \alpha(y) \), then
\[
\eta_X(x) \leq \alpha(x) \leq \alpha(y).
\]
Conversely, \( \eta_X(x) \leq \alpha(y) \) implies that
\[
\alpha(x) = \alpha^T \cdot \eta_X(x) \leq \alpha^T \cdot \alpha(y) \leq \alpha(y).
\]

3.5 Proposition. Kleisli morphisms \( f : (X, \alpha) \to (Y, \beta) \) are monotone.
Proof. Suppose that $x, y \in X$ are such that $x \leq y$, or equivalently $\alpha(x) \leq \alpha(y)$. One has
\[
\eta_Y \cdot f(x) = Tf \cdot \eta_X(x) \leq Tf \cdot \alpha(x) \leq Tf \cdot \alpha(y) \leq \beta \cdot f(y).
\]
Lemma 3.4 then implies that $\beta \cdot f(x) \leq \beta \cdot f(y)$, so that $f(x) \leq f(y)$ by definition of the order on $Y$.

3.6 Proposition. If $\mathbb{T}$ is an order-adjoint monad and $f : (Y, \beta) \to (X, a^*)$ is a Kleisli morphism with $a : TX \to X$ an Eilenberg-Moore algebra structure, then
\[
f = a \cdot Tf \cdot \beta.
\]
In particular, if $f : (Y, \beta) \to (TY, \mu_Y^*)$ is a Kleisli morphism, then
\[
f = f^\mathbb{T} \cdot \beta.
\]
Proof. On one hand, one has
\[
a \cdot Tf \cdot \beta \leq a \cdot a^* \cdot f = f,
\]
while on the other,
\[
f = a \cdot \eta_Y \cdot f = a \cdot Tf \cdot \eta_Y \leq a \cdot Tf \cdot \beta,
\]
so the claimed equality holds.

3.7 Proposition. For any map $f : X \to Y$, one has that $Tf : (TX, \mu_X^*) \to (TY, \mu_Y^*)$ is a morphism of $\mathbb{T}$-monoids, and $(Tf)^* : (TY, \mu_Y^*) \to (TX, \mu_X^*)$ is one whenever the monad $\mathbb{T}$ is enhanced. In particular, the extension operation $(-)^\mathbb{T}$ sends Kleisli morphisms to Kleisli morphisms.

Proof. The morphism $Tf : (TX, \mu_X^*) \to (TY, \mu_Y^*)$ is the image of $f : X \to Y$ by the left adjoint to the forgetful functor $\text{Set}^{\mathbb{T}} \to \text{Set}$ (that sends $f : X \to Y$ to $Tf : (TX, \mu_X) \to (TY, \mu_Y)$) followed by the functor $Q : \text{Set}^{\mathbb{T}} \to \text{Set}(\mathbb{T})$ of Example 3.3(4).

If $\mathbb{T}$ is enhanced, then $Tf \cdot (Tf)^* \leq 1_{TY}$ yields
\[
Tf \cdot \mu_X \cdot T(Tf)^* = \mu_Y \cdot TTf \cdot T(Tf)^* \leq \mu_Y \cdot T1_{TY} = \mu_Y.
\]
By exploiting the appropriate adjunction, this inequality implies in turn that
\[
T(Tf)^* \cdot \mu_Y^* \leq \mu_X^* \cdot (Tf)^*,
\]
so $(Tf)^*$ is also a Kleisli morphism. The last claim of the proposition simply follows from the fact that for $h : (X, \alpha) \to (TY, \gamma)$, one has $h^\mathbb{T} = \mu_Y \cdot Th$, that is, $h^\mathbb{T}$ is the composite of two Kleisli morphisms, and is one in turn.
3.8 **Initial structures.** In the context of Kleisli monoids, a set $X$ is equipped with the *initial structure* $\alpha$ induced by the map $f : X \to (Y, \beta)$ (in which case $f : (X, \alpha) \to (Y, \beta)$ becomes an *initial* Kleisli morphism) when the following condition is verified.

If $h : (Z, \gamma) \to (Y, \beta)$ is a Kleisli morphism and $g : Z \to Y$ a map making the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
\downarrow{h} & & \downarrow{\beta} \\
X & \xrightarrow{f} & Y
\end{array}
\]

commute, then $g : (Z, \gamma) \to (X, \alpha)$ is a Kleisli morphism.

If such initial liftings $g$ exist for all maps $f : X \to (Y, \beta)$, the forgetful functor $R : \text{Set}(\mathbb{T}) \to \text{Set}$ becomes a *fibration*.

3.9 **Example.** An initial morphism of Kleisli monoids is given by $\eta_X : (X, \eta_X) \to (TX, \mu^*_X)$. Indeed, if $g : (Y, \beta) \to (X, \eta_X)$ is such that $\eta_X \cdot g : (Y, \beta) \to (TX, \mu^*_X)$ is a Kleisli morphism, then

\[
Tg \cdot \beta = \mu_X \cdot T(\eta_X \cdot g) \cdot \beta \leq \mu_X \cdot \mu^*_X \cdot (\eta_X \cdot g) = \eta_X \cdot g .
\]

Proposition 3.12 shows that this is not an isolated occurrence when the monad $\mathbb{T}$ is enhanced.

3.10 **Proposition.** Whenever $\mathbb{T}$ is an enhanced order-adjoint monad, the forgetful functor $R : \text{Set}(\mathbb{T}) \to \text{Set}$ is a fibration. The initial structure on $X$ induced by $f : X \to (Y, \beta)$ is given by

\[
\alpha := (Tf)^* \cdot \beta \cdot f .
\]

**Proof.** Let us first verify that $\alpha$ is the structure of a Kleisli monoid. As $Tf \cdot \eta_X = \eta_Y \cdot f \leq \beta \cdot f$, one obtains that

\[
\eta_X \leq (Tf)^* \cdot \beta \cdot f = \alpha .
\]

Moreover, $\mu_X \cdot T(Tf)^* \leq (Tf)^* \cdot \mu_Y$ by adjunction of the Kleisli morphism condition of $(Tf)^*$, so we can write

\[
\alpha^* \cdot \alpha = \mu_X \cdot T(\alpha) = \mu_X \cdot T(Tf)^* \cdot T\beta \cdot Tf \cdot (Tf)^* \cdot \beta \cdot f \leq (Tf)^* \cdot \mu_Y \cdot T\beta \cdot \beta \cdot f \leq (Tf)^* \cdot \beta \cdot f = \alpha ,
\]

and $(X, \alpha)$ is a Kleisli monoid. The map $f : (X, \alpha) \to (Y, \beta)$ then obviously becomes a Kleisli morphism:

\[
Tf \cdot \alpha = Tf \cdot (Tf)^* \cdot \beta \cdot f \leq \beta \cdot f .
\]

To prove that $f$ is initial, consider a Kleisli morphism $h : (Z, \gamma) \to (Y, \beta)$ and a map $g : Z \to Y$ such that $f \cdot g = h$. Since

\[
Tg \cdot \gamma \leq (Tf)^* \cdot Th \cdot \gamma \leq (Tf)^* \cdot \beta \cdot h = \alpha \cdot g ,
\]

the map $g$ is a Kleisli morphism as required. \qed

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3.11 Remark. The previous result can easily be extended to describe initial structures with respect to more general sources: in the presence of a family of maps \((f_i : X \to Y_i)_{i \in I}\) into \(T\)-monoids \((Y_i, \beta_i)\) with \(T\) enhanced, if the infimum
\[
\alpha := \bigwedge_{i \in I} (Tf_i)^* \cdot \beta_i \cdot f_i
\]
eexists in the pointwise ordered hom-set \(\text{Set}(X, TX)\), then \((f_i : (X, \alpha) \to (X, \beta_i))_{i \in I}\) is an initial source in \(\text{Set}(\mathbb{T})\). In particular, when all sets \(TX\) are complete lattices, the category of Kleisli monoids is topological over \(\text{Set}\).

3.12 Proposition. If \(\mathbb{T}\) is enhanced, then the structure morphism \(\alpha : X \to TX\) of a \(\mathbb{T}\)-monoid \((X, \alpha)\) satisfies
\[
\alpha = (\alpha^T)^* \cdot \alpha
\]
Therefore, \(\alpha : X \to TX\) is the initial structure on \(X\) induced by \(\alpha : X \to (TX, \mu_X^*)\).

Proof. The inequality \(\eta_X \leq \alpha\) implies \(1_{TX} = \eta_X^T \leq \alpha^T\), so that \((\alpha^T)^* \leq \alpha^T \cdot (\alpha^T)^* \leq 1_{TX}\).

Therefore,
\[
\alpha \leq (\alpha^T)^* \cdot \alpha \leq \alpha,
\]
that is, \(\alpha = (\alpha^T)^* \cdot \alpha\), as required. Since \((\alpha^T)^* \cdot \alpha = (T\alpha)^* \cdot \mu_X^* \cdot \alpha\), the last statement then follows from Proposition 3.10. \(\Box\)

4 A derived Kock-Zöberlein monad

4.1 Construction. Starting from an order-adjoint monad \(\mathbb{T} = (T, \eta, \mu)\), we outline the construction of a Kock-Zöberlein monad \(\mathbb{T}'\) on the category \(\text{Set}(\mathbb{T})\) of Kleisli monoids. Essentially, the components of the monad \(\mathbb{T}\) can be restricted to the set \(T'X\) of elements of \(TX\) that are invariant under \(\alpha^T\). We first detail the construction itself of the monad \(\mathbb{T}'\), while the proofs that these definitions are adequate are deferred to Lemma 4.2 and Proposition 4.3 below.

(i) For a Kleisli monoid \((X, \alpha)\), one defines \(T'X\), the set of \(\alpha^T\)-invariants, as the (tacitly chosen) equalizer in \(\text{Set}\) of the pair \((\alpha^T, 1_{TX})\):
\[
\begin{array}{c}
T'X \xrightarrow{s_X} TX \\
\downarrow \alpha^T \quad \downarrow 1_{TX}
\end{array}
\]

As an equalizer of an idempotent and the identity, the map \(s_X\) is a section. More precisely, there exists a map \(r_X : TX \to T'X\) such that
\[
s_X \cdot r_X = \alpha^T \quad \text{and} \quad r_X \cdot s_X = 1_{T'X}.
\]
Indeed, by definition of a Kleisli monoid, one has \(\alpha^T \cdot \alpha = \alpha\), so that
\[
\alpha^T \cdot \alpha = (\alpha^T \cdot \alpha)^T = \alpha^T;
\]
the universal property of pullbacks yields the existence of a unique $r_X : TX \to T'X$ making the following diagram commute:

\[
\begin{array}{ccc}
TX & \xrightarrow{\alpha^T} & TX \\
\downarrow{r_X} & \quad & \downarrow{1_{TX}} \\
T'X & \xrightarrow{s_X} & TX
\end{array}
\]

Therefore, one obtains $s_X \cdot r_X = \alpha^T$, the first equality in (†). The previous displayed equation obviously yields that $\alpha^T \cdot \alpha^T \cdot s_X = \alpha^T \cdot s_X$, so there is a unique map $f : T'X \to T'X$ such that the diagram

\[
\begin{array}{ccc}
T'X & \xrightarrow{f} & TX \\
\downarrow{s_X} & \quad & \downarrow{1_{TX}} \\
T'X & \xrightarrow{s_X} & TX
\end{array}
\]

commutes. Since both $1_{T'X}$ and $r_X \cdot s_X$ are suitable candidates for $f$, we conclude that $r_X \cdot s_X = 1_{TX}$, the second equality of (†).

The set $T'X$ can then be equipped with the structure $\omega_X : T'X \to TT'X$ defined by

\[
\omega_X := Tr_X \cdot \mu_X \cdot s_X.
\]

(ii) One has $\alpha^T \cdot \alpha = \alpha$, so there exists a unique map $\eta'_X : X \to T'X$ making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & TX \\
\downarrow{\eta'_X} & \quad & \downarrow{1_{TX}} \\
T'X & \xrightarrow{s_X} & TX
\end{array}
\]

(iii) If $(Y, \beta)$ is a Kleisli monoid, and $f : (X, \alpha) \to (T'Y, \omega_Y)$ a Kleisli morphism, then one has

\[
\beta^T \cdot (s_Y \cdot f)^\top = (\beta^T \cdot s_Y \cdot f)^\top = (s_Y \cdot f)^\top.
\]

Thus, there exists a unique map $f'^T : T'X \to T'Y$ making the following diagram commute:

\[
\begin{array}{ccc}
T'X & \xrightarrow{(s_Y \cdot f)^\top} & s_X \\
\downarrow{f'^T} & \quad & \downarrow{1_{T'Y}} \\
T'Y & \xrightarrow{\beta^T} & TY
\end{array}
\]

4.2 Lemma. For a Kleisli monoid $(X, \alpha)$, the map $\omega_X : T'X \to TT'X$ defined in (i) is a Kleisli monoid structure that is moreover the initial structure on $T'X$ induced by $s_X : T'X \to TX$. As a consequence, $s_X$ is an equalizer in $\text{Set}(\mathbb{T})$, and the maps $\eta'_X$ and $f'^T$ defined in (ii) and (iii) are Kleisli morphisms.
Proof. To verify that \( \omega_X \) is a Kleisli monoid structure, observe that (\( \dag \)) implies

\[
\eta_{TX} = \eta_{TX} \cdot r_X \cdot s_X = T_{TX} \cdot \eta_{TX} \cdot s_X \leq T_{TX} \cdot \mu_X^* \cdot s_X = \omega_X ;
\]

the second condition follows from the fact that

\[
T_{TX} \cdot T\alpha \cdot \mu_X^* \leq T_{TX} \cdot \mu_X^* \cdot T\alpha \leq \mu_X^* \cdot \mu_X \cdot T\alpha
\]

since one then has

\[
\omega_X \cdot \omega_X = \mu_{TX} \cdot TTr_X \cdot T(\mu_X^*) \cdot T_{TX} \cdot T\alpha \cdot \mu_X^* \cdot s_X
\]

\[
= T_{TX} \cdot \mu_{TX} \cdot T(\mu_X^*) \cdot T_{TX} \cdot T\alpha \cdot \mu_X^* \cdot s_X
\]

\[
\leq T_{TX} \cdot \mu_{TX} \cdot T(\mu_X^*) \cdot T\alpha \cdot T\alpha \cdot s_X
\]

\[
= T_{TX} \cdot \mu_X^* \cdot \mu_X \cdot T\alpha \cdot s_X
\]

\[
\omega_X
\]

(\( \alpha^T \cdot s_X = s_X \)).

One also has that \( s_X : (T'X, \omega_X) \to (TX, \mu_X^* ) \) is a Kleisli morphism, since \( s_X \cdot r_X = \alpha^T \) and the equation (\( \star \)) above allow us to write

\[
T_{sX} \cdot \omega_X = T_{sX} \cdot T_{TX} \cdot \mu_X^* \cdot s_X = T_{TX} \cdot T\alpha \cdot \mu_X^* \cdot s_X \leq \mu_X^* \cdot \mu_X \cdot T\alpha \cdot s_X = \mu_X^* \cdot s_X .
\]

Suppose now that \( f : (Y, \beta) \to (TX, \mu_X^*) \) is a Kleisli morphism and \( g : Y \to T'X \) is a map such that \( f = s_X \cdot g \). Since \( r_X \cdot s_X = 1_{T'X} \), one has

\[
Tg \cdot \beta = Tr_X \cdot T_{sX} \cdot g \cdot \beta = Tr_X \cdot Tf \cdot \beta \leq Tr_X \cdot \mu_X^* \cdot s_X \cdot g = \omega_X \cdot g
\]

and \( g \) is a Kleisli morphism. This proves that \( s_X : (T'X, \omega_X) \to (TX, \mu_X^* ) \) is indeed an initial morphism, a fact that readily yields the last claims of the statement. \( \square \)

4.3 Proposition. The construction detailed in (i)–(iii) above defines a Kleisli triple \( (T', \eta', (\cdot)^T') \) on \( \text{Set}(\mathbb{T}) \).

Proof. Lemma 4.2 shows that the components of the triple are well-defined, so we are left to prove the conditions (\( \star \)) of 2.1. Referring to the diagram (\( \dag \)) in (iii), we have for a Kleisli morphism \( f = \eta_X' : (X, \alpha) \to (T'X, \omega_X) \) that

\[
(s_X \cdot \eta_X')^T \cdot s_X = \alpha^T \cdot s_X = s_X
\]

so the unique map \( T'X \to T'X \) is the identity \( 1_{T'X} \), as required: \( (\eta_X')^T = 1_{T'X} \). Given now any Kleisli morphism \( f : (X, \alpha) \to (T'Y, \omega_Y) \), we observe that

\[
(s_Y \cdot f)^T \cdot s_X \cdot \eta_X' = (s_Y \cdot f)^T \cdot \alpha = s_Y \cdot f
\]

(by Proposition 3.6), and we can deduce that \( f^T \cdot \eta_X' = f \) by unicity of the induced map in the diagram (\( \dag \)). Finally, consider another Kleisli morphism \( g : (Y, \beta) \to (T'Z, \omega_Z) \), with \( (Z, \gamma) \) a Kleisli monoid. The map \( (g^T \cdot f)^T \) is induced by \( (s_Z \cdot g, f)^T \cdot s_X \), and we have

\[
((s_Z \cdot g)^T \cdot f)^T \cdot s_X = ((s_Z \cdot g)^T \cdot s_Y \cdot f)^T \cdot s_X = (s_Z \cdot g)^T \cdot (s_Y \cdot f)^T \cdot s_X = s_Z \cdot g^T \cdot f^T .
\]

Therefore, by unicity of the induced map, we can conclude that \( (g^T \cdot f)^T = g^T \cdot f^T . \) \( \square \)
4.4 Proposition. Given a Kleisli monoid \((X, \alpha)\), the initial preorder on \(T'X\) induced by \(\omega_X : T'X \to TT'X\) is an order that moreover makes \(s_X : T'X \to TX\) into an order-embedding, and \(r_X : TX \to T'X\) into a monotone map. If \(T\) is enhanced, then \(r_X : (TX, \mu_X^*) \to (T'X, \omega_X)\) is a Kleisli morphism and the pair \((r_X, s_X)\) forms an adjoint situation \(r_X \dashv s_X\).

Proof. Consider the initial preorder on \(T'X\) induced by \(\omega_X\). Let us first verify that \(s_X\) is an order-embedding. Since

\[\mu_X \cdot Ts_X \cdot \omega_X = \mu_X \cdot T\mu_X \cdot TT\alpha \cdot \mu_X^* \cdot s_X = \mu_X \cdot T\alpha \cdot \mu_X \cdot \mu_X^* \cdot s_X = s_X,\]

one has for \(x, y \in T'X\) that

\[x \leq y \iff \omega_X(x) \leq \omega_X(y) \iff s_X(x) \leq s_X(y),\]

so that \(s_X\) is indeed an order-embedding. If \(x \leq y\) and \(y \leq x\), then \(s_X(x) = s_X(y)\) (as \(TX\) is an ordered set), so \(r_X = r_X \cdot s_X(x) = r_X \cdot s_X(y) = y\) by point (i) of 4.1; therefore, \(T'X\) is indeed an ordered set. Thus, for \(x, y \in TX\) one has that

\[r_X(x) \leq r_X(y) \iff s_X \cdot r_X(x) \leq s_X \cdot r_X(y) \iff \alpha^T(x) \leq \alpha^T(y).\]

As \(\alpha^T\) is monotone by the hypotheses on \(T\), one concludes that \(x \leq y \implies r_X(x) \leq r_X(y)\), that is, \(r_X\) is also monotone. In the case where \(T\) is enhanced, one has \(1_{TX} = \eta^T_X \leq \alpha^T\), so

\[Tr_X \cdot \mu_X^* \leq Tr_X \cdot \mu_X^* \cdot \alpha^T = Tr_X \cdot \mu_X^* \cdot s_X \cdot r_X = \omega_X \cdot r_X,\]

which proves that \(r_X : (TX, \mu_X^*) \to (T'X, \omega_X)\) is a \(T\)-monoid morphism. Finally, the two inequalities

\[1_{TX} = \eta^T_X \leq \alpha^T = s_X \cdot r_X \quad \text{and} \quad r_X \cdot s_X = 1_{T'X}\]

yield that \((r_X, s_X)\) forms an adjoint situation. \(\square\)

4.5 Theorem. There is an isomorphism of Eilenberg-Moore categories that is moreover identical on morphisms:

\[\text{Set}^T \cong \text{Set}(T)^{T'}.\]

Proof. Consider first a \(T\)-algebra \((X, a)\). The functor \(Q : \text{Set}^T \to \text{Set}(T)\) (see Example (4) of 3.3) yields a Kleisli monoid \((X, a^*)\), which can be equipped with the structure \(a' : (T'X, \omega_X) \to (X, a^*)\) defined by

\[a' := a \cdot s_X.\]

Let us first verify that \(a'\) is indeed a morphism of Kleisli monoids:

\[Ta' \cdot \omega_X = Ta \cdot Ts_X \cdot Tr_X \cdot \mu_X^* \cdot s_X \quad \text{(definitions of } a' \text{ and } \omega_X)\]
\[= T(a \cdot \mu_X \cdot T(a^*)) \cdot \mu_X^* \cdot s_X \quad (s_X \cdot r_X = \mu_X \cdot T(a^*))\]
\[= Ta \cdot \mu_X^* \cdot s_X \quad (a \cdot \mu_X = a \cdot Ta \text{ and } a \cdot a^* = 1_X)\]
\[\leq a^* \cdot a \cdot s_X \quad (Ta \cdot \mu_X^* \leq a^* \cdot a)\]
\[= a^* \cdot a'.\]
Moreover, $a'$ also satisfies the two Eilenberg-Moore conditions. For the first, observe that
\[ a' \cdot \eta' X = a \cdot s_X \cdot \eta' X = a \cdot a^* = 1_X \]
by definition of $\eta' X$, and Lemma 2.6. For the second condition, we use the Kleisli triple characterization (see 2.2): if $f, g : (Y, \beta) \to (T' X, \omega_X)$ are Kleisli morphisms with $a' \cdot f = a' \cdot g$, then we have $a \cdot s_X \cdot f = a \cdot s_X \cdot g$, so that $a \cdot (s_X \cdot f)^\top = a \cdot (s_X \cdot g)^\top$; therefore
\[ a' \cdot f^\top = a \cdot s_X \cdot f^\top = a \cdot (s_X \cdot f)\top \cdot s_Y = a \cdot (s_X \cdot g)\top \cdot s_Y = a \cdot s_X \cdot g^\top = a' \cdot g^\top , \]
by exploiting the definition of the extension operation $(-)^\top$. We conclude that $((X, a^*), a \cdot s_X)$ is indeed a $T'$-algebra. If $f : (X, a) \to (Y, b)$ is a morphism of Eilenberg-Moore algebras, then it is a Kleisli morphism $f : (X, a^*) \to (Y, b^*)$. Moreover, since $b$ is an Eilenberg-Moore algebra structure, one first observes that
\[ b \cdot (b^* \cdot f)^\top = b \cdot \mu_X \cdot (b^*) \cdot T f = b \cdot T b \cdot T (b^*) \cdot T f = b \cdot T f \]
by Lemma 2.6. To verify that $b' \cdot (\eta'_Y \cdot f)^\top = f \cdot a'$, we use the previous observation in
\[ b' \cdot (\eta'_Y \cdot f)^\top = b \cdot s_Y \cdot (\eta'_Y \cdot f)^\top = b \cdot (s_Y \cdot \eta'_Y \cdot f)^\top \cdot s_X = b \cdot (b^* \cdot f)^\top \cdot s_X = f \cdot a \cdot s_X = f \cdot a' . \]
Therefore, $f : ((X, a^*), a \cdot s_X) \to ((Y, b^*), b \cdot s_Y)$ is a morphism of Eilenberg-Moore algebras.

Suppose now that a $T'$-algebra $((X, \alpha), a')$ is given. The structure map $a : TX \to X$ is defined by
\[ a := a' \cdot \tau_X . \]

We may write
\[ a \cdot \eta_X = a' \cdot \tau_X \cdot \eta_X = a' \cdot \tau_X \cdot s_X \cdot \tau_X \cdot \eta_X = a' \cdot \tau_X \cdot a^\top \cdot \eta_X = a' \cdot \tau_X \cdot \alpha = a' \cdot \eta'_X = 1_X . \]

Consider maps $f, g : Y \to TX$ such that $a \cdot f = a \cdot g$. Thus, the Kleisli morphisms $r_X \cdot f, r_X \cdot g : (Y, \eta_Y) \to (T' X, \omega_X)$ satisfy $a' \cdot r_X \cdot f = a' \cdot r_X \cdot g$, so that $a' \cdot (r_X \cdot f)^\top = a' \cdot (r_X \cdot g)^\top$. Since $s_Y \cdot r_Y = \eta'_Y = 1_{TX}$ and $s_X \cdot r_X = \alpha^\top$, one has
\[ a \cdot f^\top = a' \cdot r_X \cdot s_X \cdot r_X \cdot f^\top \\
= a' \cdot r_X \cdot a^\top \cdot f^\top \]
\[ = a' \cdot r_X \cdot (a^\top \cdot f)^\top \\
= a' \cdot r_X \cdot (s_X \cdot r_X \cdot f)^\top \cdot s_Y \cdot r_Y \\
= a' \cdot (r_X \cdot f)^\top \cdot r_Y . \]

Therefore, $a \cdot f^\top = a' \cdot (r_X \cdot f)^\top \cdot r_Y = a' \cdot (r_X \cdot g)^\top \cdot r_Y = a \cdot g^\top$, as required. Thus, $(X, a' \cdot r_X)$ is an Eilenberg-Moore algebra. Let $f : ((X, \alpha), a') \to ((Y, \beta), b')$ be a morphism of Eilenberg-Moore algebras. Since $\eta_Y$ is the identity for Kleisli composition, one obtains that
\[ r_Y \cdot \eta_Y = r_Y \cdot s_Y \cdot r_Y \cdot \eta_Y = r_Y \cdot \beta^\top \cdot \eta_Y = r_Y \cdot \beta = \eta'_Y . \]
Exploiting this, and proceeding as in the penultimate displayed equation above, we get

\[ b \cdot T f = b \cdot (\eta_Y \cdot f)^T = b' \cdot (\eta_Y \cdot f)^T \cdot r_X = b' \cdot (\eta_Y \cdot f)^T \cdot r_X = f \cdot a' \cdot r_X = f \cdot a \cdot r_X = f \cdot a. \]

This proves that \( f : (X, a' \cdot r_X') \rightarrow (Y, b' \cdot r_Y) \) is a morphism of Eilenberg-Moore algebras.

The previous paragraphs define functors \( \overline{Q} : \text{Set}^\mathbb{T} \rightarrow \text{Set}(\mathbb{T})^\mathbb{T}' \) and \( \overline{R} : \text{Set}(\mathbb{T})^\mathbb{T}' \rightarrow \text{Set}^\mathbb{T} \) that are both identical on morphisms. It easily follows that they are inverse of one another. Indeed, we first observe that for a \( \mathbb{T}' \)-algebra \((X, \alpha), a')\), one has \( a' \cdot r_X \cdot \alpha = a' \cdot \eta_X = 1_X \); one also has \( Ta' \cdot \omega_X \leq \alpha \cdot a' \) because the structure \( a' \cdot r_X \) is an \( \mathbb{T} \)-monoid morphism; hence,

\[ 1_{T_X} = Ta' \cdot Tr_X \cdot T\alpha \leq Ta' \cdot \omega_X \cdot r_X \leq \alpha \cdot a' \cdot r_X. \]

As \( a' \cdot r_X \) and \( \alpha \) are both monotone, one concludes that \((a' \cdot r_X)^* = \alpha\). This means that the corresponding \( \mathbb{T} \)-algebra \((X, a' \cdot r_X)\) gives back the \( \mathbb{T}' \)-algebra \((X, \alpha)\) with structure \( a' \cdot r_X \cdot s_X = a' \).

Conversely, starting from a \( \mathbb{T} \)-algebra \((X, a)\), we obtain the Kleisli monoid structure \( \alpha = a^* \) and

\[ a \cdot s_X \cdot r_X = a \cdot a^* \cdot a = a \cdot \mu_X \cdot T(a^*) = a \cdot Ta \cdot T(a^*) = a \]

by Lemma 2.6. This proves that \( \overline{Q} \overline{R} = 1_{\text{Set}(\mathbb{T})^{\mathbb{T}'}} \) and \( \overline{R} \overline{Q} = 1_{\text{Set}^\mathbb{T}}. \)

**4.6 Corollary.** If \( R : \text{Set}(\mathbb{T}) \rightarrow \text{Set} \) denotes the functor that forgets the structure of objects, then the maps \( r_X \) form the components of a natural transformation \( r : TR \rightarrow RT' \), and the pair \((R, r) : \mathbb{T} \rightarrow \mathbb{T}' \) defines a monad morphism.

*Proof.* The proof of Theorem 4.5 defines a functor \( \overline{R} : \text{Set}(\mathbb{T})^{\mathbb{T}'} \rightarrow \text{Set}^\mathbb{T} \) that is algebraic over \( R : \text{Set}(\mathbb{T}) \rightarrow \text{Set} \). The discussion in 2.2 then yields the natural transformation \( \sigma : TR \rightarrow RT' \) for which \((R, \sigma)\) forms a monad morphism. Explicitly, the \( \overline{R} \)-image of \(((T'X, \omega_X), 1_{T'X}^{\mathbb{T}'})\) is \((T'X, 1_{T'X}^{\mathbb{T}'} \cdot r_{T'X})\), and the component of \( \sigma \) at \((X, \alpha)\) is obtained as

\[
\sigma_X = 1_{T'X}^{\mathbb{T}'} \cdot r_{T'X} \cdot T\eta_X',
\]

\[
= r_X \cdot s_X \cdot r_{T'X} \cdot \tau_{T'X} \cdot \eta_{T'X} \cdot \eta_X',
\]

\[
= r_X \cdot s_X \cdot \omega_X \cdot (\eta_{T'X} \cdot \eta_X')^T
\]

\[
= r_X \cdot ((s_X^T \cdot \omega_X)^T \cdot \eta_{T'X} \cdot \eta_X')^T
\]

\[
= r_X \cdot (s_X^T \cdot \omega_X \cdot \eta_X')^T
\]

\[
= r_X \cdot (s_X \cdot \eta_X')^T
\]

\[
= r_X
\]

(Proposition 3.6)

\[
(r_X) \cdot a^* = r_X \cdot s_X \cdot r_X.
\]

**4.7 Corollary.** The monad \( \mathbb{T}' \) restricts to the category \( \text{Set}(\mathbb{T})_0 \) of separated Kleisli monoids, and the isomorphism of Theorem 4.5 becomes:

\[ \text{Set}^\mathbb{T} \cong \text{Set}(\mathbb{T})_0^{\mathbb{T}'} \].
4.8 Properties of the derived monad. Let us recall that in the presence of an order-adjoint monad $\mathbb{T}$ the sets $T'X$ defined in 4.1 are equipped with the initial order induced by $\omega_X : T'X \to TT'X$ (see Proposition 4.4). More generally, the underlying set $X$ of a $\mathbb{T}$-monoid $(X, \alpha)$ inherits the preorder on $TX$ via

$$x \leq y \iff \alpha(x) \leq \alpha(y).$$

These preorders make the components $\eta'_X$ and $\mu'_X$ monotone, as well as all $T'f : (T'X, \omega_X) \to (T'Y, \omega_Y)$ that come from Kleisli morphisms $f : (X, \alpha) \to (Y, \beta)$. Indeed, Proposition 4.4 states that both $r_X$ and $s_X$ are monotone, and we can write

$$\eta'_X = r_X \cdot \alpha,$$

$$\mu'_X = 1_{T'X} = r_X \cdot s_X \cdot s_{T'X},$$

$$T'f = (\eta'_X \cdot f)^T = r_Y \cdot (\beta \cdot f)^T \cdot s_X = r_Y \cdot T f \cdot s_X,$$

the last equality coming from the fact that

$$r_Y \cdot (\beta \cdot f)^T = r_Y \cdot (\beta^T \cdot \eta'_X \cdot f)^T = r_Y \cdot \beta^T \cdot (\eta'_Y \cdot f)^T = r_Y \cdot s_Y \cdot r_Y \cdot T f = r_Y \cdot T f.$$

When the extension operation $(-)^T$ preserves the order on hom-sets $\text{Set}(X, TY)$, one has for Kleisli morphisms $f, g : (X, \alpha) \to (Y, \beta)$ that

$$f \leq g \implies T'f \leq T'g$$

(with $\text{Set}(\mathbb{T})(X, Y)$ pointwise preordered). Since every Kleisli morphism is monotone with respect to the preorders induced by the structure morphisms (Proposition 3.5), $\mathbb{T}'$ is in fact a monad on the preorder-enriched category $\text{Set}(\mathbb{T})$ and if $\mathbb{T}$ is enhanced, then $\mathbb{T}'$ is a 2-functor.

4.9 Theorem. For every $\mathbb{T}$-monoid $(X, \alpha)$, one has $\mu'_X \cdot \eta'_{TX} = 1_{T'X}$. Therefore, if $\mathbb{T}$ is an enhanced order-adjoint monad, then its derived monad $\mathbb{T}'$ on $\text{Set}(\mathbb{T})$ is Kock-Zöberlein.

Proof. Since $\mathbb{T}'$ is a monad, one readily has $\mu'_X \cdot \eta'_{TX} = 1_{T'X}$ (where $(X, \alpha)$ is a Kleisli monoid). Moreover,

$$s_{T'X} \leq T r_X \cdot \mu'^*_X \cdot \mu_X \cdot T s_X \cdot s_{T'X}$$

$$= T r_X \cdot \mu'^*_X \cdot s_X \cdot s_{T'X}$$

$$= T r_X \cdot \mu'^*_X \cdot s_X \cdot \mu'_X$$

$$= \omega_X \cdot \mu'_X$$

$$= s_{T'X} \cdot \eta'_{T'X} \cdot \mu'_X.$$
As $s_{T'}X$ is an order-embedding by Proposition 4.4, one has that $1_{T'T'}X \leq \eta_{T'T'}X \cdot \mu'_{T'}X$, and there is an adjoint situation $\mu'_{T'}X \vdash \eta_{T'T'}X$. If $T$ is enhanced, then $T'$ is a 2-functor, so that $\mu'_{T'}X \vdash \eta_{T'T'}X$ yields that $T'$ is Kock-Zöberlein (see 2.3).

4.10 Corollary. If $T$ is an enhanced order-adjoint monad, then the monad $T'$ restricted to $\text{Set}(T)_{0}$ is Kock-Zöberlein.

Proof. Corollary 4.7 states that $T'$ does indeed restrict to $\text{Set}(T)_{0}$, and it obviously remains Kock-Zöberlein.

4.11 Corollary. In the case where $T$ is an enhanced order-adjoint monad, a Kleisli monoid $(X,\alpha)$ is of the form $(X,a^*)$ for a $T$-algebra $(X,a)$ if and only if $\alpha$ is a reflective embedding, that is, if and only if $\alpha$ has a left adjoint $\alpha^* : TX \to X$ satisfying $\alpha^* \cdot \alpha = 1_{TX}$.

Proof. Suppose first that $\alpha = a^*$ for a $T$-algebra $(X,a)$. Since $a \cdot \alpha = 1_{X}$ by Lemma 2.6, $\alpha$ is a reflective embedding.

Conversely, if $\alpha$ is a reflective embedding, it has a left adjoint $\alpha^*$ such that $\alpha^* \cdot \alpha = 1_{X}$ (notice that $X$ also becomes an ordered set with the order induced by $\alpha : X \to TX$). Since $\eta'_{X} = r_{X} \cdot \alpha$, one has for $a' := \alpha^* \cdot s_{X}$ that

$$1_{T'T'}X \leq r_{X} \cdot \alpha \cdot \alpha^* \cdot s_{X} = \eta'_{X} \cdot a' \quad \text{and} \quad a' \cdot \eta'_{X} = \alpha^* \cdot s_{X} \cdot r_{X} \cdot \alpha = \alpha^* \cdot \alpha = 1_{X}$$

so that $a' \vdash \eta'_{X}$. As $T'$ is Kock-Zöberlein by Theorem 4.9, one obtains a $T'$-algebra $(X,a')$ that yields in turn a $T$-algebra $(X,a)$ with $a = a' \cdot r_{X}$ by the correspondence described in the proof of Theorem 4.5. After observing that $a \vdash \alpha$, one can conclude that $\alpha = a^*$, as claimed.

5 Injective Kleisli monoids.

5.1 Right adjoints. In 4.8, we demonstrate that the monad $T'$ inherits many of the properties of the original $T$, and, enjoys new ones of its own. However, right adjoints of the form $(T'f)^*$ for Kleisli morphisms $f : (X,\alpha) \to (Y,\beta)$ were not discussed. The obvious candidate for the right adjoint is $(T'f)^* : T'Y \to T'X$ given by the monotone map

$$(T'f)^* := r_{X} \cdot (Tf)^* \cdot s_{Y}.$$

In fact, in the case where $T$ is enhanced, the inequalities

$$1_{T'T'}X \leq (T'f)^* \cdot T'f \quad \text{and} \quad T'f \cdot (T'f)^* \leq 1_{T'T'}Y$$

do indeed hold, and $(T'f)^*$, being a composite of Kleisli morphisms (Proposition 3.7), is also one. Let $\text{Ini}$ denote the class of all initial Kleisli morphisms, and $\text{Emb}$ the class of all embeddings, that is, of all initial Kleisli morphisms whose underlying maps are monomorphisms (see [1]). Our goal now is to describe the $M$-injective Kleisli monoids with $M = \text{Ini}$ or $M = \text{Emb}$, that is, algebras
(X, α) such that, given morphisms of Kleisli monoids \( f : (Y, \beta) \to (X, \alpha) \) and \( j : (Y, \beta) \to (Z, \gamma) \) with \( j \in M \), there exists a morphism \( \overline{f} : (Z, \gamma) \to (X, \alpha) \) that extends \( f \) along \( j \):

\[
\begin{array}{ccc}
(Y, \beta) & \xrightarrow{j} & (Z, \gamma) \\
\downarrow{f} & & \downarrow{\overline{f}} \\
(X, \alpha) & & (X, \alpha)
\end{array}
\]

Given an order-adjoint monad \( T \), the category of \( M \)-injective Kleisli monoids with those Kleisli morphisms that have a right adjoint Kleisli morphism is denoted by \( M\text{-Inj}(\text{Set}(T)) \), while its full subcategory whose objects are the separated Kleisli monoids is denoted by \( \text{Ini}^{-}\text{Inj}(\text{Set}(T)_0) \).

Naturally, if there is an isomorphism such as \( \text{Set}(T) \cong A \), we also use the corresponding notation \( M\text{-Inj}(A) \).

**5.2 Lemma.** If \( T \) is enhanced, then for a morphism of Kleisli monoids \( f : (X, \alpha) \to (Y, \beta) \), one has

\[
\alpha = (Tf)^* \cdot \beta \cdot f \iff \eta_X' = (Tf)^* \cdot \eta_Y' \cdot f .
\]

As a consequence, the structure morphism \( \alpha : X \to TX \) of a Kleisli monoid \((X, \alpha)\) is the initial structure on \( X \) induced by the map \( \eta_X' : X \to (T'X, \omega_X) \).

**Proof.** Suppose that \( \alpha = (Tf)^* \cdot \beta \cdot f \) holds. Then

\[
(Tf)^* \cdot \eta_Y' \cdot f = r_X \cdot (Tf)^* \cdot s_Y \cdot \eta_Y' \cdot f = r_X \cdot (Tf)^* \cdot \beta \cdot f = r_X \cdot \alpha = \eta_X' .
\]

Conversely, if \( \eta_X' = (Tf)^* \cdot \eta_Y' \cdot f \), then, recalling that \( Tf \cdot r_X = r_Y \cdot Tf \) by Corollary 4.6, and that \( s_X = r_X^* \) by Proposition 4.4, we can write

\[
\alpha = s_X \cdot \eta_X' = r_X^* \cdot (Tf)^* \cdot \eta_Y' \cdot f = (r_Y \cdot Tf)^* \cdot \eta_Y' \cdot f = (Tf)^* \cdot s_Y \cdot \eta_Y' \cdot f = (Tf)^* \cdot \beta \cdot f .
\]

Consider now the map \( \eta_X' : X \to (T'X, \omega_X) \). The initial structure induced on \( X \) by \( \eta_X' \) is the initial structure on \( X \) induced by \( \alpha = s_X : \eta_X' : X \to (TX, \mu_X^*) \) (see Proposition 10.45 in [1]). But the latter is given by \( \alpha \) itself (Proposition 3.12), so our proof is complete.

**5.3 Theorem.** If \( T \) is enhanced, then there is an isomorphism of categories

\( \text{Ini}^{-}\text{Inj}(\text{Set}(T)) \cong \text{Set}(T')^{\text{T}} \)

that is identical on morphisms. In particular, the Ini-injective \( T \)-monoids are exactly the \( T' \)-algebras.

**Proof.** Consider a Kleisli monoid \((X, \alpha)\) that is an Ini-injective object. By Lemma 5.2, the Kleisli morphism \( \eta_X' : (X, \alpha) \to (T'X, \omega_X) \) is initial, so there is a Kleisli morphism \( a' : (T'X, \omega_X) \to (X, \alpha) \) that extends \( 1_X : (X, \alpha) \to (X, \alpha) \) along \( \eta_X' \):

\[
\begin{array}{ccc}
(X, \alpha) & \xrightarrow{\eta_X'} & (T'X, \omega_X) \\
\downarrow{1_X} & & \downarrow{a'} \\
(X, \alpha) & & (X, \alpha)
\end{array}
\]

\(\ast\)
that is, such that \( a' \cdot \eta'_X = 1_X \). Moreover, by using that \( T' \) is Kock-Zöberlein and \( \eta' \) a natural transformation, we get

\[
1_{T'X} = T'a' \cdot T' \eta'_X \leq T'a' \cdot \eta'_T = \eta'_X \cdot a'.
\]

In fact, one has \( a' \cdot \eta'_X \) as \( a' \) is also monotone (Proposition 3.5). Since \( \text{Set}(T) \) is preorder-enriched and \( T' \) is a Kock-Zöberlein monad, we have that \( a' \cdot T'a' \cong a' \cdot \mu'_X \) (that is, \( a' \cdot T'a' \leq a' \cdot \mu'_X \) and \( a' \cdot \mu'_X \leq a' \cdot T'a' \)); but this equivalence is induced by the order on \( T'X \), and is therefore an equality \( a' \cdot T'a' = a' \cdot \mu'_X \); thus, \( (X, a') \) is a \( T' \)-algebra. Let us briefly come back to the definition of \( a' \): the adjoint situation \( a' \cdot \eta'_X \) suggests that the morphism \( a' \) is determined up to equivalence in \( X \) (which is a preordered set in general); but \( a' \cdot r_X \cdot \alpha = 1_X \) implies that the preorder on \( X \) is antisymmetric, so that \( a' \) is really uniquely determined.

Suppose that \( f : (X, \alpha) \to (Y, \beta) \) is a morphism between \( \text{Inj} \)-injective Kleisli monoids that has a right adjoint \( f^* : (Y, \beta) \to (X, \alpha) \) which is also a Kleisli morphism. The previous construction yields Eilenberg-Moore algebras \( (X, a') \) and \( (Y, b') \), respectively. One then has

\[
b' \cdot T'f = b' \cdot r_Y \cdot Tf \cdot s_X \\
\leq b' \cdot r_Y \cdot Tf \cdot s_X \cdot \eta'_X \cdot a' \quad (a' \cdot \eta'_X) \\
\leq b' \cdot r_Y \cdot \beta \cdot f \cdot a' \quad (Tf \cdot \alpha \leq \beta \cdot f) \\
= f \cdot a' \quad (b' \cdot r_Y \cdot \beta = b' \cdot \eta'_Y = 1_Y),
\]

that is, \( b' \cdot T'f \leq f \cdot a' \). In the same way, one obtains \( a' \cdot T'(f^*) \leq f^* \cdot b' \), which is equivalent to \( f \cdot a' \leq b' \cdot T'f \), and we can conclude that \( f : (X, a') \to (Y, b') \) is a morphism of Eilenberg-Moore algebras.

Conversely, any \( T' \)-algebra \( ((X, \alpha), a') \) makes the diagram (*) above commute. Thus, given any initial morphism \( j : (Y, \beta) \to (Z, \gamma) \), and morphism \( f : (Y, \beta) \to (X, \alpha) \), one can define \( \overline{f} := a' \cdot T'f \cdot (T'j)^* \cdot \eta'_Z \) (as suggested in [5]):

\[
\begin{array}{ccc}
(T'Y, \omega_X) & \xrightarrow{(T'j)^*} & (T'Z, \omega_Z) \\
\downarrow & & \downarrow \eta'_Z \\
(T'f) & \xrightarrow{\overline{f}} & (X, \alpha)
\end{array}
\]

The morphism \( \overline{f} : (Z, \gamma) \to (X, \alpha) \) does indeed extend \( f \) along \( j \), since by Lemma 5.2 and Proposition 3.10 one has

\[
\overline{f} \cdot j = a' \cdot T'f \cdot (T'j)^* \cdot \eta'_Z \cdot j = a' \cdot T'f \cdot \eta'_Y = a' \cdot \eta'_X \cdot f = f.
\]

This proves that \( (X, \alpha) \) is an \( \text{Inj} \)-injective object of \( \text{Set}(T) \).

If \( f : ((X, \alpha), a') \to ((Y, \beta), b') \) is a morphism of Eilenberg-Moore algebras, the Kleisli morphism \( f^* : (Y, \beta) \to (X, \alpha) \) defined by

\[
f^* = a' \cdot r_X \cdot (Tf)^* \cdot \beta
\]

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is right adjoint to $f$. Indeed, one readily verifies that the inequalities
\[ 1_X \leq f^* \cdot f \quad \text{and} \quad f \cdot f^* \leq 1_Y \]
hold, and $f^*$ is monotone by Proposition 3.5.

The passages from $Ini$-injective Kleisli monoids to Eilenberg-Moore algebras and back described above obviously define functors identical on morphisms and that are inverse of one another. This allows us to conclude the proof of the theorem.

5.4 Corollary. If $T$ is enhanced, then there is an isomorphism that is identical on morphisms between the category of $Emb$-injective separated Kleisli monoids (with left adjoint morphisms) and the category of $T'$-algebras ($T'$ can be seen here as a monad on $\mathbf{Set}(T)_0$):
\[ Emb-Inj(\mathbf{Set}(T)_0) \cong (\mathbf{Set}(T)_0)^{T'} \]

Proof. The proof that an $Ini$-injective object is an Eilenberg-Moore category in Theorem 5.3 relies on the fact that the structure morphism $\alpha : X \to TX$ of a Kleisli monoid $(X, \alpha)$ belongs to the class of initial morphisms. By definition, the structure morphisms of separated Kleisli monoids have such underlying maps, so the same proof yields the stated isomorphism.

5.5 Remark. Theorem 5.3 was strongly motivated by Theorem 4.2.2 of [5] in which Escardó presents a characterization of the Eilenberg-Moore algebras of a Kock-Zöberlein monad as $M$-injective objects, where $M$ is a class of “$T$-embeddings”. Albeit similar in spirit, the motivations for these results reveal some essential differences. On one hand, the result of op.cit. has a wider range of applications, as the Kock-Zöberlein monad involved does not need to be derived from a monad on $\mathbf{Set}$. On the other hand, initial morphisms are not “$T$-embeddings” in general; although both classes of morphisms overlap, they are different in nature. In particular, the class of “$T$-embeddings” is defined in terms of the Kock-Zöberlein monad functor $T$, while $Ini$ and $Emb$ can be defined via universal properties. For developments following [5], see in particular [17] where further references can be found.

Another strong impulse leading to Theorem 5.3 was the mention by Hofmann to the author that continuous lattices were the $Ini$-injective objects of $\mathbf{Top}$. This remark was made in the context of a discussion about the results in [9], where the reader will find further developments originating from a very different point of view.

6 Examples

In this Section, we chose a number of order-adjoint monads that appear in the literature to illustrate our main results. This list is by no way exhaustive, but—to our knowledge—the Eilenberg-Moore algebras of other instances (such as those mentioned in [6], or versions of the double-dualization monads from [12]) have not been explicitly described.
6.1 The powerset monad. The powerset monad, denoted here by \( \mathbb{P} = (P, \eta, \mu) \), has the components of its unit given by the singleton maps \( \eta_X(x) = \{ x \} \) for all \( x \in X \); the components of its multiplication are set-unions \( \mu_X(A) = \bigcup A \) for \( A \subseteq PX \). The sets \( PX \) can be ordered by set-inclusion, a structure with respect to which \( \mathbb{P} \) becomes an enhanced order-adjoint monad (in fact, \( P \) alone preserves the order on hom-sets \( Set(X, PY) \)).

The structure map of a Kleisli monoid \( (X, \alpha) \) is a map \( \alpha : X \to PX \) that can be identified with a relation \( \alpha \subseteq X \times X \) (and the Kleisli composition translates as the usual composition of relations).

The condition \( \eta_X \subseteq \alpha \) means that the relation is reflexive, and \( \alpha \circ \alpha \subseteq \alpha \) that it is transitive.

Thus, a \( \mathbb{P} \)-monoid is exactly a preordered set, and it is separated exactly when the preorder is antisymmetric. The structure map is the \textit{down-set map} of the preordered set \( X \)

\[ \alpha = \downarrow_X : X \to PX \]

that associates to an element \( x \in X \) its set of lesser elements \( \downarrow_X x = \{ y \in X \mid y \leq x \} \). A morphism \( f : (X, \alpha) \to (Y, \beta) \) of Kleisli monoids is a map that preserves the relation \( \alpha \), that is, it is a monotone map (in particular, \( \alpha \)—being a Kleisli morphism itself—is monotone, and is therefore the down-set map rather than the up-set one). One concludes that the category of \( \mathbb{P} \)-monoids is the category \( \text{PrOrd} \) of preordered sets, while the category of separated \( \mathbb{P} \)-monoids is \( \text{Ord} \):

\[ \text{Set}(\mathbb{P}) \cong \text{PrOrd} \quad \text{and} \quad \text{Set}(\mathbb{P})_0 \cong \text{Ord} \]

For a Kleisli monoid \( (X, \alpha) \), an element \( A \in PX \) is \( \alpha \)-invariant precisely when

\[ \bigcup_{x \in A} \downarrow_X x = A \]

that is, when \( A \) is \textit{down-closed}. The set \( P'X \) is therefore the set of all down-closed sets in \( X \), the map \( s_X : P'X \to PX \) is the inclusion, and \( r_X : PX \to P'X \) is the \textit{down-closure map}:

\[ r_X(A) = \bigcup_{x \in A} \downarrow x \]

for all \( A \in PX \). The \textit{down-set functor} \( P' \) then acts on monotone maps \( f : X \to Y \) as

\[ P'f(A) = \bigcup_{x \in A} \downarrow_Y f(x) \]

and one obtains the \textit{down-set monad} \( \mathbb{P}' = (P', \downarrow, \bigcup) \) on either \( \text{PrOrd} \) or \( \text{Ord} \) by Proposition 4.3 or Corollary 4.7, respectively. That \( \mathbb{P} \) is Kock-Zöberlein can of course be verified directly, but is also a consequence of Theorem 4.9.

The category of \( \mathbb{P} \)-algebras is known to be the category \( \text{Sup} \) of complete lattices with sup-preserving maps (see for example [15]). Theorem 5.3 therefore yields that complete lattices are the \textit{Ini}-injective objects of \( \text{PrOrd} \), and its Corollary 5.4 that they are also the \textit{Emb}-injective objects of \( \text{Ord} \):

\[ \text{Ini-Inj}(\text{PrOrd}) \cong \text{Sup} \cong \text{Emb-Inj}(\text{Ord}) \]

6.2 Remark. Given a monad \( T \) on \( \text{Set} \), a useful structure used to determine whether maps of the form \( Tf : TX \to TY \) as well as \( \mu_X : TTX \to TX \) have a right adjoint is the existence of a monad morphism \( \tau : \mathbb{P} \to T \). Indeed, such a morphism provides the sets \( TX \) with an order making them into complete lattices and maps \( Tf : TX \to TY \) as well as the components \( \mu_X : TTX \to TX \) into sup-maps, and all right adjoints therefore exist. The following examples are all of this form (see [20] for more details).
6.3 The filter monad. The functor $F$ of the filter monad $\mathcal{F} = (F, \eta, \mu)$ associates to a set $X$ the set $FX$ of filters on $X$ (that is, subsets of $PX$ that are closed under finite intersection and up-closure), and to a map $f : X \to Y$, the map $Ff : FX \to FY$ defined for all $B \subseteq Y$ and $x \in FX$ by

$$B \in Ff(x) \iff f^{-1}(B) \in x.$$ 

The components of the unit and multiplication can be described for $x \in X$, $A \subseteq X$, and $\mathfrak{X} \in FX$ by

$$A \in \eta_X(x) \iff x \in A \quad \text{and} \quad A \in \mu_X(\mathfrak{X}) \iff A^\mathfrak{X} \in \mathfrak{X},$$

where $A^\mathfrak{X} := \{x \in FX \mid A \in x\}$. The sets $FX$ are ordered by reverse set-inclusion (so that the monad morphism $\tau : P \to F$, with $\tau_X$ sending $A \in PX$ to its up-closure $\uparrow^P_A A \in FX$, is monotone), and make $\mathcal{F}$ into an order-adjoint monad. Since the functor $F$ preserves the pointwise order on hom-sets, the monad $\mathcal{F}$ is also enhanced.

A Kleisli monoid is a pair $(X, \alpha)$ whose structure map $\alpha : X \to FX$ associates to every point $x \in X$ a filter $\alpha(x)$ which turns out to form the neighborhood system of $x$ in an induced topology: as remarked in [6], the inequalities $\eta_X(x) \supseteq \alpha(x)$ and $\alpha^\mathfrak{X} \cdot \alpha(x) \supseteq \alpha(x)$ (for all $x \in X$) are exactly the conditions for the collection $(\alpha(x))_{x \in X}$ of filters to form a topology (see for example [3] or even [8]). A separated Kleisli monoid is then a $T_0$ topological space, and the condition for a map $f : X \to Y$ to be a Kleisli morphism $f : (X, \alpha) \to (Y, \beta)$ translates into continuity, so one can write

$$\text{Set}(\mathcal{F}) \cong \text{Top} \quad \text{and} \quad \text{Set}(\mathcal{F})_0 \cong \text{Top}_0.$$ 

Given a Kleisli monoid $(X, \alpha)$, a filter $x \in FX$ is $\alpha^\mathfrak{X}$-invariant precisely when it has a basis of open sets. Indeed, if $x = \alpha^\mathfrak{X}(\mathfrak{x})$, then

$$A \in x \iff A \in \alpha^\mathfrak{X}(x) \iff \alpha^{-1}(A^\mathfrak{X}) \in x \iff \{y \in X \mid \alpha(y) \in A\} \in x \iff A^\mathfrak{x} \in x.$$

The functor $F' : \text{Top} \to \text{Top}$ therefore sends a topological space $X$ to the set $F'X$ of filters of open sets (equipped with its Scott topology). The unit $\eta_X' : X \to F'X$ sends a point to its neighborhood filter, and $\mu_X' : F'F'X \to F'X$ is the restriction of $\mu_X$. Thus, $\mathcal{F}'$ is the open filter monad on $\text{Top}$, and Theorem 4.9 states that it is Kock-Zöberlein. It is more traditional to consider the restriction of $\mathcal{F}'$ to $\text{Top}_0$, a monad whose Kock-Zöberlein facet has been much explored (as in [5]).

By Theorem 5.3, the $\text{Ini}$-injective topological spaces are the objects of $\text{Set}^\mathcal{F}$ (or equivalently $\text{Top}^\mathcal{F}$ by Theorem 4.5). This category has been characterized (see [4], but also [25]) as the category $\text{Cnt}$ of continuous lattices and continuous sup-maps. A more classical result is that the continuous lattices form the $\text{Emb}$-injective objects of $\text{Top}_0$, a fact that follows here from Corollary 5.4. These results can be summarized by the isomorphisms:

$$\text{Ini-Inj}(\text{Top}) \cong \text{Cnt} \cong \text{Emb-Inj}(\text{Top}_0).$$

6.4 The up-set monad. The up-set monad has not enjoyed such a thorough study as the filter monad, although it enjoys many similar features. We overview some of them here, and refer to [20] for more details. The filter monad can naturally be extended to the up-set monad $\mathcal{U} = (U, \eta, \mu)$.
by “forgetting the finite intersection condition”: if \( UX \) denotes the set of all up-closed subsets of \( PX \), then one can simply replace filters by up-sets in the definition of the filter monad. For a map \( f : X \to Y \), one defines the map \( Uf : UX \to UY \), as well as the components of the unit and multiplication via

\[
B \in Uf(x) \iff f^{-1}(B) \in x, \quad A \in \eta_X(x) \iff x \in A, \quad A \in \mu_X(x) \iff A^U \in X,
\]

for all \( x \in X, A \subseteq X, B \subseteq Y, x \in UX, \mathcal{X} \in UUX \), and where \( A^U := \{x \in UX \mid A \in x\} \).

When the sets \( UX \) are ordered by reverse set-inclusion, \( U \) becomes an enhanced order-adjoint monad. The category \( \text{Set}(U) \) of Kleisli monoids then forms the category \( \text{Int} \) of interior spaces (that is, sets \( X \) equipped with an interior operator \( i : PX \to PX \)) with continuous maps, and the category \( \text{Set}(U)_0 \) is the category \( \text{Int}_0 \) of separated interior spaces:

\[
\text{Set}(U) \cong \text{Int} \quad \text{and} \quad \text{Set}(U)_0 \cong \text{Int}_0.
\]

Although the following facts might not seem surprising in view of the aforementioned results for the filter monad, to the author’s knowledge they have not appeared explicitly in the literature. Since \( \text{Set}^U \) is the category \( \text{Ccd} \) of constructive completely distributive lattices with maps that preserve simultaneously all suprema and infima (see [19]), the \( \text{Ini}-\text{injective} \) interior spaces are exactly the constructive completely distributive lattices, as are the \( \text{Emb}-\text{injective} \) separated interior spaces:

\[
\text{Ini-inj}(\text{Int}) \cong \text{Ccd} \cong \text{Emb-inj}(\text{Int}_0).
\]

6.5 The double-dualization monad. Knowing that the double-dualization functor \( D \) sends a set \( X \) to the double-powerset \( DX = PPX \), the double-dualization monad \( D = (D, \eta, \mu) \) can be described via the following three equivalences:

\[
B \in Df(x) \iff f^{-1}(B) \in x, \quad A \in \eta_X(x) \iff x \in A, \quad A \in \mu_X(x) \iff A^D \in X,
\]

for \( f : X \to Y, \mathcal{X} \in DX, x \in X, \mathcal{X} \in DDX \), and where \( A^D := \{x \in DX \mid A \in x\} \). The sets \( DX \) ordered by reverse set-inclusion make \( D \) into an order-adjoint monad, that is however not enhanced. To describe the corresponding Kleisli monoids, one exploits self-adjointness of the contravariant power-set functor: given maps \( f : X \to PY \) and \( g : PY \to PX \), one has for all \( x \in X \) and \( B \in PY \) the correspondence

\[
B \in f(x) \iff g(B) \ni x .
\]

Via this correspondence, the category of \( D \)-monoids becomes the category of “non-monotone interior spaces”, whose objects are pairs \((X, i)\), with \( i : PX \to PX \) a map satisfying

\[
i \subseteq 1_X \quad \text{and} \quad i \cdot i \subseteq i
\]

(but \( i \) not necessarily monotone), and whose morphisms \( f : (X, i_X) \to (Y, i_Y) \) are maps \( f : X \to Y \) such that

\[
f^{-1} \cdot i_X \subseteq i_Y \cdot f^{-1}.
\]
By restricting the maps $\alpha : X \to DX$ appropriately to either $\alpha : X \to UX$ or $\alpha : X \to FX$, one obtains further conditions on $i : PX \to PY$, namely that $i$ is monotone, so that $\text{Set}(U) \cong \text{Int}^{-}$ as mentioned in Example 6.4, or that $i$ preserves finite intersections, an observation that yields the isomorphism $\text{Set}(\mathcal{P}) \cong \text{Top}$ of Example 6.3. The category of $\mathbb{D}$-algebras is known to be the category $\text{CaBool}$ of complete atomistic Boolean algebras and ring homomorphisms that preserve all infima and suprema (see [16]). Theorem 4.5 therefore yields that $\text{CaBool}$ is strictly monadic over the category of “non-monotone interior spaces”, a result that is not completely surprising in view of [23].

6.6 The finitely-generated-up-set monad. If $U_{\text{fin}}X$ denotes the set of all finitely generated up-sets on $X$ (in other words, up-closures in $PPX$ of sets of finite subsets of $X$), the finitely-generated-up-set monad $U_{\text{fin}} = (U_{\text{fin}}, \eta, \mu)$ is obtained by appropriately restricting the up-set monad $U$. One has

$$A \in \eta_{X}(x) \iff x \in A, \quad A \in \mu_{X}(\mathcal{X}) \iff A^{U_{\text{fin}}} \in \mathcal{X},$$

with $A^{U_{\text{fin}}} := \{x \in U_{\text{fin}}X \mid A \subseteq x\}$, for all $x \in X$, $A \subseteq X$, and $\mathcal{X} \in U_{\text{fin}}U_{\text{fin}}X$. Similarly, if $f : X \to Y$ is a map, then $U_{\text{fin}}f : U_{\text{fin}}X \to U_{\text{fin}}Y$ is simply defined by

$$B \in U_{\text{fin}}f(x) \iff f^{-1}(B) \in x,$$

for all $B \subseteq X$ and $x \in U_{\text{fin}}X$. To make $U_{\text{fin}}$ into an order-adjoint monad, one orders the sets $U_{\text{fin}}X$ by set-inclusion. The latter become complete lattices with suprema given by union, and the maps $U_{\text{fin}}f : U_{\text{fin}}X \to U_{\text{fin}}Y$ sup-maps. This ordered structure can also be seen to be induced by the monad morphism $\sigma : \mathcal{P} \to U_{\text{fin}}$, with the components of $\sigma$ sending a subset $A \subseteq X$ to

$$\sigma_{X}(A) = \downarrow_{\mathcal{P}X}\{B \subseteq X \text{ finite} \mid A \cap B \neq \emptyset\}.$$

The one-to-one correspondence (*) described in 6.5 restricts to one between maps $f : X \to U_{\text{fin}}Y$ and monotone maps $g : PY \to PX$ that are finitary:

$$B \in g(A) \implies \text{there exists a finite subset } A' \text{ of } A \text{ with } B \in g(A').$$

Thus, the category $\text{Set}(U)$ is isomorphic to the category $\text{Cls}_{\text{fin}}$ of finitary closure spaces, that is, of sets $X$ equipped with a finitary closure operation $c : PX \to PX$, together with maps $f : (X, c_{X}) \to (Y, c_{Y})$ such that $c_{X} \cdot f^{-1} \subseteq f^{-1} \cdot c_{Y}$.

The category of $U_{\text{fin}}$-algebras forms the category $\text{Frm}$ of frames with sup-maps that preserve finite infima (see [2] or [10]), which can also be obtained (Theorem 4.5) as the category of $U_{\text{fin}}$-$\mathbb{D}$-algebras, where $U'_{\text{fin}}$ is identified with the up-set functor acting on closed subsets of the finitary closure space $(X, c)$ (the aforementioned correspondence (*) of 6.5 yields that $f \in U_{\text{fin}}X$ is $\alpha^{U_{\text{fin}}}$-invariant precisely when $A \in f \iff c(A) \in f$). Since $U_{\text{fin}}$ is enhanced, Theorem 5.3 yields the isomorphism

$$\text{Ini-Inj}(\text{Cls}_{\text{fin}}) \cong \text{Frm}.$$
6.7 The quantale-based powerset monad. A quantale (or more precisely, a unital quantale) is a complete lattice which carries a monoid structure, with monoid multiplication denoted as a tensor $\otimes$ and neutral element $k$, whose tensor distributes over suprema on both sides. Given a quantale $V$, the $V$-powerset functor $P_V$ sends a set $X$ to its $V$-powerset $V^X$, and a function $f : X \to Y$ to $P_V f : V^X \to V^Y$, where

$$P_V f(\phi)(y) := \bigvee_{x \in f^{-1}(y)} \phi(x) ,$$

for all $\phi \in V^Y$, $y \in Y$. The unit $\eta : 1_{\text{Set}} \to P_V$ and multiplication $\mu : P_V P_V \to P_V$ of the $V$-powerset monad $\mathbb{P}_V$ on $\text{Set}$ are given respectively by

$$\eta_X(x)(y) := \begin{cases} k & \text{if } x = y \\ \bot & \text{else} \end{cases} \quad \text{and} \quad \mu_X(F)(y) := \bigvee_{\phi \in V^X} F(\phi) \otimes \phi(y) ,$$

for all $x, y \in X$, $F \in V^{V^X}$, and where $\bot$ denotes the bottom element of the lattice. The corresponding extension operation $(-)^{\mathbb{P}_V}$ is defined for a map $f : X \to P_V Y$ by

$$f^{\mathbb{P}_V}(\psi)(y) := \bigvee_{x \in X} \psi(x) \otimes f(x)(y) ,$$

for all $\psi \in V^X$, $y \in Y$. Naturally, if the quantale $V$ is the two-element chain $2 = \{\bot, \top\}$, with $\otimes$ the infimum operation and $k = \top$ the top element, $\mathbb{P}_V$ is the ordinary powerset monad $\mathbb{P}$. More generally, the quantale morphism $\iota : 2 \to V$ (preserving suprema and neutral elements) induces a monad morphism $\tau : \mathbb{P} \to \mathbb{P}_V$, with $\tau_X$ sending a subset $A \subseteq X$, seen as a its characteristic function $\chi_A : X \to \{\bot, \top\}$, to the characteristic function $\tau_X(A) = \iota \cdot \chi_A$ of $A$ in $V^X$:

$$\tau_X(A)(x) := \begin{cases} k & \text{if } x \in A \\ \bot & \text{else} \end{cases}$$

As a consequence, $\mathbb{P}_V$ is an order-adjoint monad for the pointwise order on the sets $P_V X$, and one notices that $\mathbb{P}_V$ is also enhanced.

The category of $\mathbb{P}_V$-monoids can be described as follows: by identifying a map $f : X \to V^Y$ with its mate $f : X \times Y \to V$, a structure $\alpha : X \to V^X$ of a $\mathbb{P}_V$-monoid $(X, \alpha)$ becomes a map $\alpha : X \times X \to V$ such that

$$k \leq \alpha(x, x) \quad \text{and} \quad \alpha(x, y) \otimes \alpha(y, z) \leq \alpha(x, z) ,$$

for all $x, y, z \in X$. A Kleisli morphism $f : (X, \alpha) \to (Y, \beta)$ is a map $f : X \to Y$ that satisfies for all $x, y \in X$

$$\alpha(x, y) \leq \beta(f(x), f(y)) .$$

In the case where $V$ is the extended real half-line $\mathbb{R}_+ = [0, \infty]$, ordered by the opposite $\geq^\text{op}$ of the natural order $\geq$, and whose tensor is given by extended addition (so that $x + \infty = \infty + x = \infty$ for all $x \in [0, \infty]$) and $k = 0$, the category $\text{Set}(\mathbb{P}_+)$ is isomorphic to the category $\text{Met}$ whose objects are sets $X$ equipped with a generalized metric (see in particular [14]), that is, a map $\alpha : X \times X \to \mathbb{R}_+$ that satisfies

$$0 = \alpha(x, x) \quad \text{and} \quad \alpha(x, y) + \alpha(y, z) \geq \alpha(x, z) ,$$

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for all \(x, y, z \in X\), and whose morphisms \(f : (X, \alpha) \rightarrow (Y, \beta)\) are contractions, that is, maps \(f : X \rightarrow Y\) such that for all \(x, y \in X\),
\[
\alpha(x, y) \geq \beta(f(x), f(y)).
\]

In general, the underlying set \(X\) of a Kleisli monoid \((X, \alpha)\) is ordered via
\[
x \leq y \iff k \leq \alpha(x, y)
\]
for all \(x, y \in X\). Thus, it follows from Theorems 4.5 and 4.9 that \(\text{Set}^{\mathbb{P}_V}\) is the category whose objects are those separated \(\mathbb{P}_V\)-monoids \((X, \alpha)\) on complete lattices \(X\) (with respect to the order described above) whose structure \(\alpha\) is continuous in the first variable:
\[
\bigwedge_{x \in A} \alpha(x, y) = \alpha\left(\bigwedge A, y\right),
\]
for all \(y \in X\) and \(A \subseteq X\). Indeed, such a map \(\alpha : X \rightarrow \mathbb{P}_V X\) is the right adjoint of a \(\mathbb{P}_V\)-algebra structure \(a : \mathbb{P}_V X \rightarrow X\) by Corollary 4.11. The morphisms of \(\text{Set}^{\mathbb{P}_V}\) are the morphisms of \(\text{Set}(\mathbb{P}_V)\) that have a right adjoint, as in Theorem 5.3. Another description of this category can be found in [18] (for the case where \(k = \top\) in \(V\); see also [11] for the case where \(V\) is a frame, and [24] for an independent treatment), so that \(\text{Set}^{\mathbb{P}_V}\) appears as the category \(\mathbb{P}_+\text{-Mod}\) of left \(\mathbb{P}_+\)-modules, and Theorem 5.3 yields the isomorphism
\[
\text{Ini-Inj(Met)} \cong \mathbb{P}_+\text{-Mod}.
\]

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**References**


