Homological Localizations Preserve 1-Connectivity

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Abstract. Every generalized homology theory $E$ yields a localization functor $L_E$ that sends the $E$-equivalences to homotopy equivalences. We prove that if $X$ is any 1-connected space, then $L_E X$ is also 1-connected, for every generalized homology theory $E$. This is deduced from a result by Hopkins and Smith stating that if $K(\mathbb{Z}, 2)$ is $E$-acyclic then $E$ is trivial.

Introduction

A number of results in the literature suggest that idempotent functors in the homotopy category of spaces preserve 1-connectivity, although no proof of this fact has so far been given. One of the earliest examples is localization with respect to ordinary homology, which in fact preserves $n$-connectivity for all $n$; see [1].

In the same article [1], Bousfield proved the existence of localization with respect to any generalized homology theory $E$, that is, a functor $L_E$ which assigns to every space $X$ a space $L_E X$ together with a natural map $X \to L_E X$ which is terminal in the homotopy category among $E$-equivalences with source $X$. (An $E$-equivalence is a map $X \to Y$ inducing isomorphisms $E_n(X) \cong E_n(Y)$ for all $n$.)

In [9], Mislin showed that $K$-theory localization does not preserve $n$-connectivity in general, since for example $\pi_3(L_K S^{2p+2}; \mathbb{Z}/p) \neq 0$ for every odd prime $p$. However, Mislin also proved in [9] that the $K$-localization of every 1-connected space is 1-connected. Further evidence of the fact that 1-connectivity could be preserved by arbitrary idempotent functors in the homotopy category was given by Neisendorfer in [10] and by Tai in his detailed study of the problem in [11].

It is therefore natural to address the question of whether or not localizations with respect to generalized homology theories preserve 1-connectivity. Such localizations were thoroughly discussed by Bousfield in [2], where a description was given of their effect on abelian Eilenberg–Mac Lane spaces. The main tool was an arithmetic square, already exploited by Mislin in [9], allowing one to determine the $E$-localization of a space (with some restrictions on the fundamental group) from its $\mathbb{E}Z/p$-localizations and rational coherence data.
Our main result is that $L_E X$ is 1-connected if $X$ is 1-connected, for any generalization homology theory $E$. This follows by combining the methods of Bousfield in [2] with a result proved by Hopkins and Smith in [8], according to which a $K(Z, 2)$ is never $E$-acyclic if $E$ is nontrivial. We note, however, that $K(Z, 3)$ is $KZ/p$-acyclic for all $p$, by [9, Corollary 2.3]. It is known that, if $L$ is any homotopy idempotent functor, then $LK(Z, n)$ is necessarily a $K(A, n)$ where $A$ is either zero or a commutative ring with 1, for all $n$; see [5]. If $L = L_E$ for some nontrivial homology theory $E$, then the possibility that $A = 0$ has been discarded for $n = 2$ in [8], and this opens the way to substantial improvements of earlier results or to new results as in this article.

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1. Torsion homology theories

Throughout the paper we denote by $E$ a spectrum or the associated homology theory. For an abelian group $R$, the corresponding spectrum with coefficients in $R$ is defined as $ER = E \wedge SR$ where $SR$ is the Moore spectrum of type $(R, 0)$. The only cases of interest in this article are $R = Z/p$ and $R$ a subring of $Q$. A spectrum $E$ is called torsion if $EQ$ is contractible. The ordinary Eilenberg–Mac Lane spectrum with coefficients in $R$ is denoted by $HR$. We denote by $\mathbb{Z}_p^\infty$ the $p$-adic, by $\mathbb{Z}(p^\infty)$ the Prüfer group $\bigcup_{n=1}^\infty \mathbb{Z}/p^n$ and, for a set of primes $P$, we denote by $\mathbb{Z}_P$ the integers localized at $P$.

In this first section we concentrate on mod $p$ homology theories, where $p$ is any prime. Using the Atiyah–Hirzebruch spectral sequence, one sees that if $E$ is any homology theory, then every $HZ/p$-equivalence is an $EZ/p$-equivalence; details are given in [9, § 1]. Hence, all $EZ/p$-local spaces are $HZ/p$-local and there is a natural transformation of functors $\mu: L_{HZ/p} \to L_{EZ/p}$.

We next prove that, if $X$ is connected, then the induced homomorphism

$$\mu_*: \pi_1(L_{HZ/p} X) \to \pi_1(L_{EZ/p} X)$$

is surjective. This result is essentially contained in the proof of Proposition 7.1 in [2], as we next recall for the sake of completeness. The argument is based on Bousfield's version of the Whitehead theorem (cf. [2, Theorem 5.2]), stating that if $R$ is $Z/p$ or a subring of $Q$, and $f: X \to Y$ is a map inducing isomorphisms $H_i(X; R) \cong H_i(Y; R)$ for $i < n$ and an epimorphism $H_n(X; R) \to H_n(Y; R)$, where $n \geq 1$, then $f$ also induces isomorphisms $\pi_i(L_{HR} X) \cong \pi_i(L_{HR} Y)$ for $i < n$ and an epimorphism $\pi_n(L_{HR} X) \to \pi_n(L_{HR} Y)$.

**Theorem 1.1.** Let $E$ be any homology theory and $p$ any prime. Then, for every connected space $X$, the natural homomorphism $\mu_*: \pi_1(L_{HZ/p} X) \to \pi_1(L_{EZ/p} X)$ is surjective.

**Proof.** The claim is obvious if $EZ/p$ is trivial. If $EZ/p$ is not trivial, then $K(Z/p, 1)$ is not $EZ/p$-acyclic, as shown in [2, Proposition 2.2]. Since the natural map $\mu: L_{HZ/p} X \to L_{EZ/p} X$ is an $EZ/p$-equivalence, we obtain an isomorphism

$$\mu_*: H_1(L_{HZ/p} X; Z/p) \cong H_1(L_{EZ/p} X; Z/p)$$
using [2, Proposition 2.1] or [5, Theorem 1.3], according to which $K(\mathbb{Z}/p, 1)$ is $E\mathbb{Z}/p$-local. By the generalized Whitehead theorem stated above, $\mu$ induces then an epimorphism $\pi_1(L_{H\mathbb{Z}/p}X) \rightarrow \pi_1(L_{E\mathbb{Z}/p}X)$, since $L_{E\mathbb{Z}/p}X$ is $H\mathbb{Z}/p$-local. □

**Corollary 1.2.** If $E$ is any torsion homology theory and $X$ is 1-connected, then $L_E X$ is also 1-connected. □

**Proof.** As in [2], we denote by $\mathcal{P}E$ the set of primes $p$ such that $\pi_*(E)$ is not uniquely $p$-divisible. By [2, Proposition 7.1], for each torsion homology theory $E$ and every 1-connected space $X$, we have a homotopy equivalence

$$L_E X \simeq \prod_{p \in \mathcal{P}E} L_{E\mathbb{Z}/p}X.$$ 

Now recall from [1] that $L_{H\mathbb{Z}/p}X$ is 1-connected if $X$ is 1-connected. Therefore, Theorem 1.1 tells us that $L_E X$ is 1-connected. □

Before discussing non-torsion homology theories, we need to study the second homotopy group $\pi_2(L_{E\mathbb{Z}/p}X)$ when $X$ is 1-connected. The following result is of main input in our discussion.

**Theorem 1.3.** Let $E$ be a homology theory and $p$ any prime. Suppose that $E\mathbb{Z}/p$ is nontrivial. Then either $K(\mathbb{Z}/p, 2)$ or $K(\mathbb{Z}_p, 2)$ is $E\mathbb{Z}/p$-local.

**Proof.** The classification of acyclicity patterns for Eilenberg–Mac Lane spaces given by Bousfield in [2, §4] implies that $L_{E\mathbb{Z}/p}K(\mathbb{Z}, n) = K(A, n)$ for each $n \geq 1$, where the group $A$ can be $\mathbb{Z}_p^\infty$ or $\mathbb{Z}/p^i$ for some $i \geq 1$, or zero. In [8], it is shown that if a reduced homology theory vanishes on $K(\mathbb{Z}, 2)$, then it is trivial. (Thus, nontrivial mod $p$ homology theories of type IV-1 as defined in [2, §4] do not exist.) Therefore, if $L_{E\mathbb{Z}/p}$ is nontrivial, then the localization $L_{E\mathbb{Z}/p}K(\mathbb{Z}, 2)$ is necessarily $K(\mathbb{Z}_p, 2)$ or $K(\mathbb{Z}/p^i, 2)$ for some $i \geq 1$. In the latter case, $K(\mathbb{Z}/p, 2)$ cannot be $E\mathbb{Z}/p$-acyclic, as one sees by induction using the fibre sequences

$$K(\mathbb{Z}/p, 2) \rightarrow K(\mathbb{Z}/p^i, 2) \rightarrow K(\mathbb{Z}/p^{i-1}, 2).$$

Hence, $K(\mathbb{Z}/p, 2)$ is $E\mathbb{Z}/p$-local, by [2, Proposition 2.1] or [5, Lemma 1.4]. □

If $K(\mathbb{Z}/p, 2)$ is $E\mathbb{Z}/p$-local and $X$ is 1-connected, then, using the fact that $\mu : L_{H\mathbb{Z}/p}X \rightarrow L_{E\mathbb{Z}/p}X$ is an $E\mathbb{Z}/p$-equivalence, we obtain as in (1.1) an isomorphism

$$(1.2) \quad \mu_* : H_2(L_{H\mathbb{Z}/p}X; \mathbb{Z}/p) \cong H_2(L_{E\mathbb{Z}/p}X; \mathbb{Z}/p).$$

Thus, the homomorphism $\pi_2(L_{H\mathbb{Z}/p}X) \rightarrow \pi_2(L_{E\mathbb{Z}/p}X)$ induced by $\mu$ is surjective, by the generalized Whitehead theorem.

Now suppose that $K(\mathbb{Z}_p, 2)$ is $E\mathbb{Z}/p$-local and $X$ is 1-connected. Similarly as in the previous case, since $\mu$ is an $E\mathbb{Z}/p$-equivalence, we have an isomorphism

$$(1.3) \quad \mu^* : \text{Hom}(\pi_2(L_{E\mathbb{Z}/p}X), \mathbb{Z}_p) \cong \text{Hom}(\pi_2(L_{H\mathbb{Z}/p}X), \mathbb{Z}_p).$$

In order to use this information, we recall the following concept from [4, VI.3] and [7]. An abelian group $A$ is called Ext-$p$-complete if the natural homomorphism $A \rightarrow \text{Ext}(\mathbb{Z}(p^\infty), A)$ derived from the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[1/p] \rightarrow \mathbb{Z}(p^\infty) \rightarrow 0$$

is an isomorphism. Equivalently, an abelian group $A$ is Ext-$p$-complete if and only if both $\text{Hom}(\mathbb{Z}[1/p], A) = 0$ and $\text{Ext}(\mathbb{Z}[1/p], A) = 0$. As explained in [4, VI.4],
Ext-$p$-complete abelian groups are uniquely $q$-divisible for primes $q \neq p$, and they admit a canonical $\mathbb{Z}_p^\wedge$-module structure.

An Ext-$p$-complete abelian group $A$ is called adjusted if the quotient $A/T(A)$ of $A$ by its torsion subgroup $T(A)$ is $p$-divisible (hence divisible). Thus, $A$ is adjusted if and only if $A$ does not admit any torsion-free Ext-$p$-complete quotients other than zero. Since $T(A) \otimes \mathbb{Z}(p^\infty) = 0$, it also follows that an Ext-$p$-complete abelian group $A$ is adjusted if and only if $A \otimes \mathbb{Z}(p^\infty) = 0$.

**Theorem 1.4.** Let $E$ be a homology theory and $p$ a prime. Suppose that $EZ/p$ is nontrivial. Then, for every 1-connected space $X$, the cokernel of the natural homomorphism $\mu_* : \pi_2(L_{HZ/p}X) \to \pi_2(L_{EZ/p}X)$ is an adjusted Ext-$p$-complete abelian group, which is zero if $K(\mathbb{Z}/p, 2)$ is $EZ/p$-local.

**Proof.** The spaces $L_{HZ/p}X$ and $L_{EZ/p}X$ are $HZ/p$-local. The abelian groups $\pi_2(L_{HZ/p}X)$ and $\pi_2(L_{EZ/p}X)$ are Ext-$p$-complete, by [1, Theorem 5.5]. Hence, Coker $\mu_*$ is Ext-$p$-complete, since the cokernel of any homomorphism between Ext-$p$-complete abelian groups is Ext-$p$-complete. If $K(\mathbb{Z}/p, 2)$ is $EZ/p$-local, then we already proved, by means of (1.2), that Coker $\mu_*$ is zero. Thus, we assume that $K(\mathbb{Z}/p, 2)$ is $EZ/p$-local. In this case, the isomorphism displayed in (1.3) shows that $\text{Hom}(\text{Coker } \mu_*, \mathbb{Z}_p^\wedge) = 0$. For an abelian group $A$, if $\text{Hom}(A, \mathbb{Z}_p^\wedge) = 0$ then we have $\text{Hom}(A \otimes \mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) = 0$ by adjunction. Since $A \otimes \mathbb{Z}(p^\infty)$ is a $p$-torsion divisible abelian group, we may infer that $A \otimes \mathbb{Z}(p^\infty) = 0$ and this implies that $A/T(A)$ is $p$-divisible, as we needed. (In fact, an Ext-$p$-complete abelian group $A$ is adjusted if and only if the condition $\text{Hom}(A, \mathbb{Z}_p^\wedge) = 0$ holds. This has also been pointed out in [3, Lemma 7.7].) \qed

2. Non-torsion homology theories

In this section we deal with non-torsion homology theories. In this case, there is an arithmetic square allowing one to compute $E$-localizations of 1-connected spaces by combining mod $p$ data and rational data. Specifically, the following diagram is a homotopy pull-back square if $X$ is 1-connected (and also under less restrictive conditions; see [2, Proposition 7.2]). Recall that $PE$ denotes the set of primes $p$ such that $\pi_*(E)$ is not uniquely $p$-divisible.

\[
\begin{array}{ccc}
L_XX & \to & \prod_{p \in PE} L_{EZ/p}X \\
\downarrow & & \downarrow \\
L_{HQ}X & \to & L_{HQ} \left( \prod_{p \in PE} L_{EZ/p}X \right)
\end{array}
\]

We also need the following remark.

**Lemma 2.1.** Suppose given a set of primes $P$ and an adjusted Ext-$p$-complete abelian group $A_p$ for all $p \in P$. The rationalization $\prod_{p \in P} A_p \to \left( \prod_{p \in P} A_p \right) \otimes \mathbb{Q}$ is then an epimorphism.
**Proof.** Fix any prime \( q \in P \). Then we have \( A_q \otimes \mathbb{Z}(q^{\infty}) = 0 \) since \( A_q \) is adjusted, and \( \left( \prod_{p \neq q} A_p \right) \otimes \mathbb{Z}(q^{\infty}) = 0 \) as well, since \( \prod_{p \neq q} A_p \) is uniquely \( q \)-divisible. Therefore, \( \left( \prod_{p \in P} A_p \right) \otimes \mathbb{Q}/\mathbb{Z} = 0 \). This shows that \( \left( \prod_{p \in P} A_p \right) \otimes \mathbb{Q}/\mathbb{Z} = 0 \), which proves our claim. \[ \square \]

Our main result is the following.

**Theorem 2.2.** Let \( E \) be any homology theory and let \( X \) be 1-connected. Then \( L_E X \) is also 1-connected.

**Proof.** By Corollary 1.2, we may assume that \( E \) is not torsion. Our strategy is to compare the arithmetic squares for \( E \) and ordinary homology \( H_{P,E} \). The natural maps \( \mu : L_{H_{P,E}} X \to L_{E_{P,E}} X \) yield a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & F' \\
\downarrow & & \downarrow \\
F'' & \longrightarrow & \bigotimes_{p \in P} L_{H_{P,E}} X \\
\downarrow & & \downarrow \\
L_{H_{Q,P}} F'' & \longrightarrow & L_{H_{Q,P}} \left( \bigotimes_{p \in P} L_{H_{P,E}} X \right)
\end{array}
\]

where each row and each column is a fibre sequence. The four spaces in the lower right square are 1-connected by Corollary 1.2. Therefore, all the fibres except perhaps \( Y \) are connected. The group \( \pi_1(F'') \) is the product of the cokernels of the homomorphisms \( \mu_* : \pi_2(L_{H_{P,E}} X) \to \pi_2(L_{E_{P,E}} X) \), so it is a product of adjusted Ext-\( p \)-complete groups, by Theorem 1.4. Hence, Lemma 2.1 tells us that the induced homomorphism \( \pi_1(F'') \to \pi_1(L_{H_{Q,P}} F'') \) is surjective. This implies that \( Y \) is connected as well, so the homomorphism \( \pi_1(F') \to \pi_1(F) \) is surjective.

From the arithmetic square for \( E \) we see that \( L_E X \) is 1-connected if and only if the boundary homomorphism \( \pi_2(L_{H_{Q,P}} X) \to \pi_1(F) \) is surjective. Consider now the fibre sequence \( F' \to L_{H_{P,E}} X \to L_{H_{Q,P}} X \) appearing in the arithmetic square for \( H_{P,E} \). Since we know that \( L_{H_{P,E}} X \) is 1-connected, the homomorphism \( \pi_2(L_{H_{Q,P}} X) \to \pi_1(F') \) is surjective. The composite homomorphism \( \pi_2(L_{H_{Q,P}} X) \to \pi_1(F) \) is also surjective, as we needed. \[ \square \]

**References**


[6] E. Devinatz, *Hopkins' proof that \( \Sigma^\infty \mathbb{C}P^\infty \) is Bousfield equivalent to \( S^0 \),* letter, 1999.


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