Suppose that \([-, (Y, y_0)]_\ast : \text{Top}_\ast \to \text{Gr}\). Let \(\mu : Y \times Y \to Y\) be a based continuous map such that \([\mu]\ast = [pr_1]\ast \cdot [pr_2]\ast\). Prove that
\[
\mu \circ (\mu \times Id_Y) \simeq \mu \circ (Id_Y \times \mu).
\]

**Proof.** We want to show that \([\mu \circ (\mu \times Id_Y)]_\ast = [\mu \circ (Id_Y \times \mu)]_\ast\), i.e. that the following diagram commutes “homotopically”.

\[
\begin{array}{cccc}
Y \times Y \times Y & \xrightarrow{\mu \times Id_Y} & Y \times Y \\
Id_Y \times \mu & & \mu \\
Y \times Y & \xrightarrow{\mu} & Y
\end{array}
\]

Since the mapping \((\mu \times Id_Y) \in \text{Top}_\ast(Y \times Y \times Y, Y \times Y)\), under the hypothesis that the functor \([-, (Y, y_0)]_\ast\) factorizes through the category of groups \(\text{Gr}\), and by definition of a functor, we conclude that
\[
(\mu \times Id_Y)^\ast : [Y \times Y, Y]_\ast \longrightarrow [Y \times Y \times Y, Y]_\ast
\]

is a homomorphism of groups. Therefore,
\[
[\mu \circ (\mu \times Id_Y)]_\ast = (\mu \times Id_Y)^\ast([\mu]\ast) = (\mu \times Id_Y)^\ast([pr_1]\ast \cdot [pr_2]\ast)
\]
\[
= (\mu \times Id_Y)^\ast([pr_1]\ast) \cdot (\mu \times Id_Y)^\ast([pr_2]\ast)
\]
\[
= [pr_1 \circ (\mu \times Id_Y)]_\ast \cdot [pr_2 \circ (\mu \times Id_Y)]_\ast.
\]

For all \((y_1, y_2, y_3) \in Y \times Y \times Y\), we have
\[
pr_2 \circ (\mu \times Id_Y)(y_1, y_2, y_3) = pr_2(\mu(y_1, y_2), y_3) = y_3 = pr_3(y_1, y_2, y_3),
\]
thus, \([pr_2 \circ (\mu \times Id_Y)]_\ast = [pr_3]\ast\).

Now, let us define the mapping
\[
pr_{1,2} : Y \times Y \times Y \longrightarrow Y \times Y
(y_1, y_2, y_3) \longmapsto (y_1, y_2),
\]
which is clearly continuous (by universal property of the product topology).
By the same reasoning applied to \((\mu \times \text{Id}_Y)^*\), we conclude that \(pr_1^{*}\) is also a homomorphism of groups. Moreover, for all \((y_1, y_2, y_3) \in Y \times Y \times Y\), we have

\[pr_1 \circ (\mu \times \text{Id}_Y)(y_1, y_2, y_3) = pr_1(\mu(y_1, y_2), y_3) = \mu(y_1, y_2) = \mu \circ pr_1(y_1, y_2, y_3),\]

therefore,

\[\begin{align*}
[pr_1 \circ (\mu \times \text{Id}_Y)]_* &= [\mu \circ pr_1,2,*] = pr_1^{*}([\mu]_*) = pr_1^{*}([pr_1]_* \cdot [pr_2]_*) \\
&= pr_1^{*}([pr_1]_*) \cdot pr_1^{*}([pr_2]_*) = [pr_1 \circ pr_1,2,*] \cdot [pr_2 \circ pr_1,2,*] \\
&= [pr_1]_* \cdot [pr_2]_*.
\end{align*}\]

We conclude that

\[\big[\mu \circ (\mu \times \text{Id}_Y)\big]_* = [pr_1]_* \cdot [pr_2]_* \cdot [pr_3]_*.
\]

With a very similar argument we can see that \([\mu \circ (\text{Id}_Y \times \mu)]_*\) gives the same result. Explicitly, we have that \((\text{Id}_Y \times \mu)^*\) is a homomorphism of groups, therefore,

\[\begin{align*}
[\mu \circ (\text{Id}_Y \times \mu)]_* &= (\text{Id}_Y \times \mu)^*([\mu]_*) = (\text{Id}_Y \times \mu)^*([pr_1]_* \cdot [pr_2]_*) \\
&= (\text{Id}_Y \times \mu)^*([pr_1]_*) \cdot (\text{Id}_Y \times \mu)^*([pr_2]_*) \\
&= [pr_1 \circ (\text{Id}_Y \times \mu)]_* \cdot [pr_2 \circ (\text{Id}_Y \times \mu)]_*.
\end{align*}\]

For all \((y_1, y_2, y_3) \in Y \times Y \times Y\), we have

\[pr_1 \circ (\text{Id}_Y \times \mu)(y_1, y_2, y_3) = pr_1(\text{Id}_Y(y_1, y_2), y_3) = y_1 = pr_1(y_1, y_2, y_3),\]

thus, \([pr_1 \circ (\text{Id}_Y \times \mu)]_* = [pr_1]_*\). We define the continuous mapping \(pr_{2,3}\) such that \(pr_{2,3}(y_1, y_2, y_3) = (y_2, y_3)\) and obtain

\[pr_2 \circ (\text{Id}_Y \times \mu)(y_1, y_2, y_3) = pr_2(\text{Id}_Y(y_2, y_3)) = \mu(y_2, y_3) = \mu \circ pr_{2,3}(y_1, y_2, y_3).
\]

Therefore, since \(pr_{2,3}^*\) is a homomorphism of groups (same reasoning as before),

\[\begin{align*}
[pr_2 \circ (\text{Id}_Y \times \mu)]_* &= [\mu \circ pr_{2,3}]_* = pr_{2,3}^*([\mu]_*) = pr_{2,3}^*([pr_1]_* \cdot [pr_2]_*) \\
&= pr_{2,3}^*([pr_1]_*) \cdot pr_{2,3}^*([pr_2]_*) = [pr_1 \circ pr_{2,3}]_* \cdot [pr_2 \circ pr_{2,3}]_* \\
&= [pr_2]_* \cdot [pr_3]_*.
\end{align*}\]

Hence,

\[\big[\mu \circ (\text{Id}_Y \times \mu)\big]_* = [pr_1]_* \cdot [pr_2]_* \cdot [pr_3]_*.
\]

\[\Box\]