1. Let $E_*$ be a homology theory on $\text{Top}_{rel}$. Let $X_n = \{1, \ldots, n\}$, endowed with the discrete topology.

(a) Calculate $E_*(X_n, \emptyset)$ in terms of $E_*(X_1, \emptyset)$.
(b) Calculate $E_*(X_n, X_k)$ in terms of $E_*(X_1, \emptyset)$, for all $k \leq n$.
(c) For all $n \geq 1$, let $Y_n = D^1 \coprod \cdots \coprod D^n$, where $D^k$ is the disk of dimension $k$. Calculate $E_*(Y_n, Y_k)$ in terms of $E_*(X_1, \emptyset)$, for all $k \leq n$.

2. (The 5-Lemma) Let

$$
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & A_2 \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} \\
B_1 & \xrightarrow{g_1} & B_2
\end{array}
\begin{array}{ccc}
A_2 & \xrightarrow{f_2} & A_3 \\
\downarrow{\alpha_2} & & \downarrow{\alpha_3} \\
B_2 & \xrightarrow{g_2} & B_3
\end{array}
\begin{array}{ccc}
A_3 & \xrightarrow{f_3} & A_4 \\
\downarrow{\alpha_3} & & \downarrow{\alpha_4} \\
B_3 & \xrightarrow{g_3} & B_4
\end{array}
\begin{array}{ccc}
A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\alpha_4} & & \downarrow{\alpha_5} \\
B_4 & \xrightarrow{g_4} & B_5
\end{array}
$$

be a commutative diagram in $\text{Ab}$ with exact rows. Show that if

- $\alpha_2$ and $\alpha_4$ are isomorphisms,
- $\alpha_1$ is surjective, and
- $\alpha_5$ is injective,

then $\alpha_3$ is an isomorphism.

3. Let $\tau_* : E_* \to E_*'$ be a morphism of homology theories on $\text{Top}_{rel}$, i.e., $\tau_* = \{\tau_n \mid n \geq 0\}$, where, for every $n$, $\tau_n : E_n \to E_n'$ is a natural transformation such that $\tau_{n-1} \partial_n = \partial_n' \tau_n - 1$.

Apply the 5-Lemma to proving that if

$$(\tau_n)_{(X, \emptyset)} : E_n(X, \emptyset) \to E'_n(X, \emptyset)$$

is an isomorphism for all topological spaces $X$, then

$$(\tau_n)_{(X, A)} : E_n(X, A) \to E'_n(X, A)$$

is an isomorphism for all pairs $(X, A)$. 

1
4. (Split exact sequences) A short exact sequence

\[ 0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0 \]  

in $\text{Ab}$ is \textit{split} if $p$ \textit{admits a section}, i.e., if there exists a homomorphism $s : C \to B$ such that $p \circ s = \text{Id}_C$.

(a) Show that the sequence (0.1) is split if and only if $p$ admits a \textit{retraction}, i.e., there exists a homomorphism $r : B \to A$ such that $r \circ j = \text{Id}_A$.

(b) Show that the sequence (0.1) is split if and only if there is an isomorphism \( \varphi : A \oplus C \xrightarrow{\cong} B \) such that $\varphi \circ \iota = j$ and $p \circ \varphi = \pi$, where $\iota : A \hookrightarrow A \oplus C$ and $\pi : A \oplus C \to C$ are the natural inclusion and projection, respectively.

(c) Give two examples of nonsplit short exact sequences.